Measure Theory

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1 Preliminaries

1.A Limits on the Extended Real Line

Our setting for most analysis in measure theory will be the **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We review some important results regarding the infimum and supremum from real analysis, adapted to this new setting.

Definition: Infimum and Supremum

Let $S \subseteq \mathbb{R}$ be a nonempty set.

The **infimum** (or greatest lower bound) of S, denoted $\inf S$, is the largest number m such that $m \leq s$ for all $s \in S$. Equivalently:

(i) $m \leq s$ for all $s \in S$ (lower bound)
(ii) For any $\varepsilon > 0$, there exists $s \in S$ with $s < m + \varepsilon$ (greatest)

The **supremum** (or least upper bound) of S, denoted $\sup S$, is the smallest number M such that $s \leq M$ for all $s \in S$. Equivalently:

(i) $s \leq M$ for all $s \in S$ (upper bound)
(ii) For any $\varepsilon > 0$, there exists $s \in S$ with $s > M - \varepsilon$ (least)

We set $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$ by convention.

Remark: Characterization of Infimum in $\overline{\mathbb{R}}$

Now that we are in $\overline{\mathbb{R}}$, the characterization of the infimum and supremum changes a little bit:

Given $Y \subseteq [0, +\infty]$ nonempty, $\inf(Y) \in \begin{cases} \{+\infty\} \text{ if } Y = \{+\infty\} \\ [0, +\infty) \text{ otherwise} \end{cases}$

In the former case, $\forall \varepsilon > 0$, $\inf Y = \inf Y + \varepsilon = +\infty$, and there does not exist $y \in Y$ such that $y < \inf Y + \varepsilon$.

In the latter case, $\forall \varepsilon > 0$, $\inf Y < \inf Y + \varepsilon$, so $\inf Y + \varepsilon$ is **not** a lower bound of Y and there exists $y \in Y$ such that $y < \inf Y + \varepsilon$.

Definition: Limit Superior and Limit Inferior

Given a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$, define

$$\limsup_{n \to +\infty} x_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} x_n \quad \text{and} \quad \liminf_{n \to +\infty} x_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} x_n.$$

Proposition

Given a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$, we have $\liminf_{n \to +\infty} x_n \leq \limsup_{n \to +\infty} x_n$.

Proof: Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$. Consider the sequences $a_k := \inf_{n\geq k} x_n$ and $b_k := \sup_{n\geq k} x_n$, for a fixed k.

Now, it is clear that $a_k \leq b_k \forall k \in \mathbb{N}$, implying $\lim_{k\to+\infty} a_k \leq \lim_{k\to+\infty} b_k$.

We claim that a_k is nondecreasing, since

$$a_{k+1} = \inf_{n\geq k+1} x_n \geq \min\left(x_k, \inf_{n\geq k+1} x_n\right) = \inf_{n\geq k} x_n = a_k$$

Similarly, b_k is nonincreasing, since

$$b_{k+1} = \sup_{n\geq k+1} x_n \leq \max\left(x_k, \sup_{n\geq k+1} x_n\right) = \sup_{n\geq k} x_n = b_k.$$

Thus by the Monotone Convergence Theorem, we must have that

$$\lim_{k\to+\infty} a_k = \sup_{k\in\mathbb{N}} a_k \quad \text{and} \quad \lim_{k\to+\infty} b_k = \inf_{k\in\mathbb{N}} b_k.$$

Thus

$$\liminf_{n\to+\infty} x_n = \sup_{k\in\mathbb{N}} a_k = \lim_{k\to+\infty} a_k \leq \lim_{k\to+\infty} b_k = \inf_{k\in\mathbb{N}} b_k = \limsup_{n\to+\infty} x_n.$$

□

Proposition

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$,

$$\limsup_{n\to+\infty}(-x_n) = -\liminf_{n\to+\infty} x_n.$$

Proof: Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$. Observe that

$$\begin{aligned}
-\liminf_{n\to+\infty} x_n &= -\sup_{k\in\mathbb{N}}\left(\inf_{n\geq k} x_n\right) && \text{definition} \\
&= \inf_{k\in\mathbb{N}}\left(-\inf_{n\geq k} x_n\right) && -\sup(S) = \inf(-S) \\
&= \inf_{k\in\mathbb{N}}\left(\sup_{n\geq k}(-x_n)\right) && -\inf(S) = \sup(-S) \\
&= \limsup_{n\to+\infty}(-x_n) && \text{definition}
\end{aligned}$$

□

Proposition

For any sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \in \overline{\mathbb{R}}$,

$$\limsup_{n\to+\infty}(x_n + y_n) \leq \limsup_{n\to+\infty} x_n + \limsup_{n\to+\infty} y_n,$$

as long as none of the sums are of the form $\infty - \infty$.

Proof: Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$. Observe that, for a fixed $n \geq k$,

$$x_n \leq \sup_{j\geq k} x_j \text{ and } y_n \leq \sup_{j\geq k} y_j$$

so, as long as we have no indeterminate expression of the form $\infty - \infty$, we have

$$x_n + y_n \leq \sup_{j\geq k} x_j + \sup_{j\geq k} y_j.$$

Now since the supremum is the least upper bound,

$$\sup_{n\geq k}(x_n + y_n) \leq \sup_{j\geq k} x_j + \sup_{j\geq k} y_j$$

Thus taking limits gives

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

using the same argument as in one of the previous propositions: the limit of a nonincreasing sequence is the same as the infimum of that sequence.

□

Example: Strict Inequality Applies

An example where the strict inequality holds in the previous proposition is $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then observe $\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty}((-1)^n - (-1)^n) = \limsup_{n\to\infty} 0 = 0$. But $\limsup_{n\to\infty} (-1)^n = 1 = \limsup_{n\to\infty} (-1)^{n+1}$, which can be shown easily through subsequential limits.

Proposition

If $x_n \leq y_n$ for all n,

$$\liminf_{x\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n$$

Proof: Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{R}}$. Since $x_n \leq y_n$, there must exist some k so that $n \geq k \implies \inf_{n\geq k} x_n \leq \inf_{n\geq k} y_n$. Then we can take the supremum over $k \in \mathbb{N}$ on both sides, giving the result:

$$\sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \liminf_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} y_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} y_n$$

□

Theorem

For any real valued sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$,

$$\boxed{x_n \text{ converges} \iff \limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n}$$

Furthermore, if either equivalent condition holds, then $x_* = \limsup_{n\to+\infty} x_n = \liminf_{n\to+\infty} x_n$ is the limit of x_n.

1.B Topology

Definition: Topology and Topological Space

A **topology** τ on X is a collection of subsets of X that

(i) Contains $\emptyset$ and X

(ii) Is closed under arbitrary unions

(iii) Is closed under finite intersections

We call the pair (X, τ) a **topological space**.

Definition: Open and Closed Set

Elements of a topology τ are called **open sets**. Complements of open sets are called **closed sets**.

Definition: Topological Convergence

On a topological space X, a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ converges to a limit $x \in X$ if, for any open set U containing x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Definition: Compact Set

A topological space X is called **compact** if for every collection C of open subsets of X such that

$$X = \bigcup_{S \in C} S,$$

there is a finite subcollection $F \subseteq C$ such that

$$X = \bigcup_{S \in F} S.$$

Note that the collection C is called an **open cover** and the finite collection F is called a **finite subcover**.

Theorem: Heine-Borel Theorem

Given $S \subseteq \mathbb{R}^n$,

$$S \text{ is compact} \iff S \text{ is closed and bounded}$$

1.C The Riemann Integral

Recall that the **Riemann integral** is the formalization of approximation the area under the graph of a function by using approximating rectangles. In particular, for a nonnegative function $f : [a, b] \to \mathbb{R}$,

$$\int_a^b f(x)\,\mathrm{d}x$$

should be the area of the set

$$S = \{(x, y) : x \in [a, b], 0 \leq y \leq f(x)\}.$$

If f changes sign, we write $f = f_+ - f_-$, where

$$f_+(x) = \max\{f(x), 0\}; \quad f_-(x) = -\min\{f(x), 0\},$$

with

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f_+(x)\,\mathrm{d}x - \int_a^b f_-(x)\,\mathrm{d}x\,.$$

We can formally define it as follows. Fix an interval $[a, b]$, $a \neq b$.

Definition: Partition

A partition P of $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$ satisfying

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Define $\Delta x_i = x_i - x_{i-1}$. For any bounded, real valued function $f : [a, b] \to \mathbb{R}$, we may define the **upper and lower sums** with respect to a given partition P:

Definition: Upper and Lower Sums

$$U(P, f) := \sum_{i=1}^{n} M_i \Delta x_i, \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),$$
$$L(P, f) := \sum_{i=1}^{n} m_i \Delta x_i, \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

This leads to the definition of the **upper and lower Riemann integrals**:

Definition: Riemann Integrable

Define the **upper and lower Riemann integrals** of f over $[a, b]$ by

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf_P U(P, f)$$
$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup_P L(P, f).$$

If $\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b}$, then we say f is **Riemann integrable** on $[a, b]$, and the value of its integral is given by

$$\boxed{\int_a^b f(x)\,\mathrm{d}x := \overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x\,.}$$

1.D Limitations of the Riemann Integral

Riemann integration is nice, but falls short in a couple areas. For example, it struggles to handle weird sets and limits.

Exercise: A Function where Riemann Integration Fails

Let $f : [0, 1] \to \mathbb{R}$ be the function that is 1 for every rational number and 0 for every irrational number. Prove that f is not Riemann integrable on $[0, 1]$.

Proof: Let P be a partition $x_0, x_1, ..., x_n$ of $[0, 1]$ with

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1.$$

Now examine a particular interval $[x_{i-1}, x_i)$. Since the rationals are dense in the reals, $\exists q \in [x_{i-1}, x_i) \cap \mathbb{Q}$, implying that $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \geq 1$. But f is also bounded above by 1, so we have $M_i = 1$. But since the irrationals are also dense in the reals, $\exists w \in [x_{i-1}, x_i) \cap (\mathbb{R} \setminus \mathbb{Q})$, implying that $m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x) \leq 0$. And since f is bounded below by 0 we have $m_i = 0$.

Now since i was arbitrary, we have

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1$$
$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = 0$$

But now since our partition was arbitrary, we also have

$$\overline{\int_0^1} f(x)\,\mathrm{d}x = \inf_P U(P, f) = 1$$
$$\underline{\int_0^1} f(x)\,\mathrm{d}x = \sup_P L(P, f) = 0$$

so the upper and lower Riemann integrals are different, showing that f is not Riemann integrable.

□

1.E Motivation for Measure Theory

More generally, one of the most important questions that we seek to answer in real analysis is the following: Given $f_n : [a, b] \to \mathbb{R}$ with $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} f_n(x) = f(x),$$

when can we prove that

$$\lim_{n \to \infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x?$$

Measure theory greatly expands the tools we can use to answer this question.

2 Introduction to Measures

2.A Problems with the Naive Definition

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Consider a set $E \subseteq \mathbb{R}$. For reasons that will become apparent later, it would be nice to have a way to describe the "total size" of that set. In particular, we would like to define a function $\mu : 2^{\mathbb{R}^d} \to [0, +\infty]$ (where 2^X for a set X denotes the power set of X) such that:

Concept: Ideal Properties of a Measure

(i) We assign the "right size" to simple sets. For example, $\mu([a,b]) = b - a$ for $a \leq b$, and in particular $\mu([a,a]) = 0$.

(ii) If $\{E_i\}_{i=1}^n \subseteq 2^{\mathbb{R}^d}$ are disjoint, then

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i),$$

a property which makes μ **finitely additive**. We can extend this to the notion of being **countably additive**:

$$\{E_i\}_{i=1}^\infty \subseteq 2^{\mathbb{R}^d} \text{ disjoint} \implies \mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$

(iii) μ is **translation invariant**. This means that for all $E \subseteq \mathbb{R}^d$ and $c \in \mathbb{R}^d$, we have $\mu(E + c) = \mu(E)$, where we define $E + c = \{x + c : x \in E\}$.

Unfortunately for analysts, there is no such function satisfying each of the properties above, a fact proved by Giuseppe Vitali in 1905. Before proceeding with the proof of the theorem, we proceed with a quick lemma showing monotonocity of finitely additive measures:

Lemma: Monotonicity of Finitely Additive Measures

Given a set X and a finitely additive function $\mu : 2^X \to [0, +\infty]$, then $\forall A, B \in X$, we have

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

Proof: Observe

$$\begin{aligned} \mu(B) &= \mu(A \cup (B \setminus A)) \\ &= \mu(A) + \mu(B \setminus A) \\ &\geq \mu(A). \end{aligned}$$

□

Now onto the main result:

Theorem: Vitali Theorem

There does not exist any function $\mu : 2^{\mathbb{R}^d} \to [0, +\infty]$ that satisfies
(i) $\mu([a, b]) = b - a$ for $a \leq b$
(ii) Countable additivity
(iii) Translation invariance

Proof: Assume by contradiction that such a μ exists.

Define an equivalence relation on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$, with equivalence classes $[x] = \{y \in \mathbb{R} : y \sim x\}$.

We claim that every equivalence class contains an element in $[0, 1]$.

Proof: Take some $x \in \mathbb{R}$ and denote its equivalence class by $[x]$. Let $y = x - \lfloor x \rfloor$. Notice that $y \in [x]$ since $x - (x - \lfloor x \rfloor) = \lfloor x \rfloor \in \mathbb{Z} \subseteq \mathbb{Q}$. Further, $y \in [0, 1]$ since $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1 \Rightarrow 0 \leq x - \lfloor x \rfloor \leq 1$. □

Thus for each equivalence class, we can choose an element in $[0, 1]$ belonging to that class. Let A be the set of elements chosen (note that we used the Axiom of Choice to construct this set.)

Now define

$$B = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (A + q).$$

We claim this is a disjoint union.

Proof: Let $U, V \in \{A + q\}_{q \in \mathbb{Q} \cap [-1,1]}$ with $U \neq V$. So $\exists q_1, q_2 \in \mathbb{Q} \cap [-1, 1]$ such that $U = A + q_1$ and $V = A + q_2$ with $q_1 \neq q_2$. By contradiction suppose $t \in U \cap V$. Then $\exists a_1, a_2 \in A$ such that $t = a_1 + q_1$ and $t = a_2 + q_2$, so $a_1 - a_2 = q_2 - q_1$.

Now observe we must have $a_1 \neq a_2$, because otherwise $q_1 = q_2$, contradicting $U \neq V$. Now since a_1 and a_2 are in different equivalence classes (since there is only one representative from each), we know $a_1 - a_2 \notin \mathbb{Q}$. But since $\mathbb{Q}$ is closed under addition, we also know $q_2 - q_1 \in \mathbb{Q}$. Thus we have a contradiction. □

We claim that $[0, 1] \overset{(i)}{\subseteq} B \overset{(ii)}{\subseteq} [-1, 2]$.

- To see inclusion (ii), observe $A \subseteq [0, 1]$ and $\mathbb{Q} \cap [-1, 1] \subseteq [-1, 1]$, so $a + q \in [-1, 2] \forall a \in A, q \in \mathbb{Q} \cap [-1, 1]$.
- To see inclusion (i), observe given $x \in [0, 1]$, we have $x \in [a]$ for some $a \in A$ by our earlier claim. Thus $x - a = q$ for some $q \in \mathbb{Q}$, and since $a \in [0, 1]$ and $x \in [0, 1]$, we have $q \in \mathbb{Q} \cap [-1, 1]$, showing $x \in B$.

Now, by the previous lemma and function property (i), we must have

$$1 = \mu([0, 1]) \leq \mu(B) \leq \mu([-1, 2]) = 3.$$

But we also have

$$\begin{aligned}
\mu(B) &= \mu\left(\bigcup_{q \in \mathbb{Q} \cap [-1,1]} (A + q)\right) && \text{definition} \\
&= \sum_{q \in \mathbb{Q} \cap [-1,1]} \mu(A + q) && \text{property (ii), since the union is disjoint} \\
&= \sum_{q \in \mathbb{Q} \cap [-1,1]} \mu(A) && \text{property (iii)} \\
&= \sum_{i=1}^{\infty} \mu(A) && \text{since } \mathbb{Q} \text{ is countably infinite}
\end{aligned}$$

Now since $\mu(B) \leq 3$, we must have $\mu(A) = 0$. But then $\mu(B) = 0$, contradicting $1 \leq \mu(B)$.

□

This result tells us that we must weaken at least one of our criteria to get a function with the desired properties.

We might consider loosening property (i). But this is the property that inspired the whole concepts of "measures", and if we change it, we might lose all notion of length and volume. The same goes for property (iii): we want to maintain the intuition that translating length leaves it unchanged. Thus, we resolve to modify property (ii).

Notice that Vitali sets broke down our function by taking advantage of countably infinite sets. So, a natural attempt to fix this is to replace countable additivity with the weaker notion of finite additivity.

However, this fails too, when we try higher dimensions.

Theorem: Banach-Tarski Paradox (1924)

Given any two bounded subsets with nonempty interior A and B of $\mathbb{R}^d$, with $d \geq 3$, there exist partitions of A and B into a finite number of disjoint subsets

$$A = \bigcup_{i=1}^{n} A_i, \quad A_i \cap A_j = \emptyset, i \neq j,$$

and

$$A = \bigcup_{i=1}^{n} B_i, \quad B_i \cap B_j = \emptyset, i \neq j,$$

such that for each $i \in \{1, 2, ..., k\}$, the sets A_i and B_i are congruent. That is, one is obtained from the other through translations, reflections and rotations in $\mathbb{R}^d$.

This theorem shows, for example, that using the notion of volume given by this naive properties, we can show that a baseball and the moon have the same volume.

Thus, let's reduce the number of sets we consider from being the entire powerset of $\mathbb{R}^d$, so that we can prevent these pathological problems from popping up.

2.B σ-Algebras

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Then the question becomes: what subsets of $\mathbb{R}^d$ might we want to measure?

One way to think about this question is to just assume that we have some existing collection of "measurable" sets, and think about them as building blocks. If we can measure two sets, it's natural to want to be able to measure their union, intersection, and complements, since these are the most important operations we perform on sets. So, let's define some structure to encode this idea.

Definition: Algebra of Sets

Let X be a nonempty set and let $\mathcal{A} \subseteq 2^X$ be a family of subsets of X. We say that $\mathcal{A}$ is an **algebra of sets** if

(i) $\{E_i\}_{i=1}^n \subseteq \mathcal{A} \Longrightarrow \cup_{i=1}^n E_i \in \mathcal{A}$ (closure under finite unions)
(ii) $E \in \mathcal{A} \Longrightarrow E^c \in \mathcal{A}$ (closure under complements)

Notice that we didn't bother to include closure under finite intersections, since it follows from the above two properties, as the next proposition shows:

Proposition

If $\mathcal{A}$ is an algebra of sets of X, then
(i) $\{E_i\}_{i=1}^n \subseteq \mathcal{A} \Longrightarrow \cap_{i=1}^n E_i \in \mathcal{A}$
(ii) $\emptyset, X \in \mathcal{A}$

Proof:

(i) Suppose $E_1, ..., E_n \in \mathcal{A}$. By closure under complements, we have $E_1^c, ..., E_n^c \in \mathcal{A}$. Since $\mathcal{A}$ is closed under countable unions, we have $\cup_{i=1}^n E_i^c \in \mathcal{A}$. Then since $\mathcal{A}$ is closed under complements, we have $(\cup_{i=1}^n E_i^c)^c = \cap_{i=1}^n E_i \in \mathcal{A}$.

(ii) Let $E \in \mathcal{A}$. Then by closure of complements, we have $E^c \in \mathcal{A}$. Thus $E \cup E^c = X \in \mathcal{A}$. By the first part, we also have $E \cap E^c = \emptyset \in \mathcal{A}$.

□

Example

(i) $\mathcal{A} = 2^X$
(ii) $\mathcal{A} = \{\emptyset, X\}$
(iii) $\mathcal{A}$ is the set of clopen sets in any topology
(iv) $\mathcal{A}$ is the collection of finite and cofinite subsets of X

In analysis, we very often deal with limits, so it would be nice to restrict to those algebras which are closed under countable unions. This leads us to a very important type of algebra, called a **σ-algebra**.

Definition: σ-algebra

$\mathcal{A} \subseteq 2^X$ is a **σ-algebra** of subsets of X if
(i) $\mathcal{A}$ is an algebra
(ii) $\mathcal{A}$ is closed under countable unions:

$$\{E_i\}_{i=1}^\infty \subseteq \mathcal{A} \Longrightarrow \cup_{i=1}^\infty E_i \in \mathcal{A}$$

Note that this means that showing closure under complements and countable unions is sufficient to show a set is a σ-algebra, since we can take sets in the union after a certain N to be the empty set.

Remark

A σ-algebra is also closed under countable intersections, which follows from closure under countable unions and using De Morgan's Law.

Remark

Some key differences between a σ-algebras and topologies:
(i) σ-algebras are closed under countable unions while topologies are closed under any unions
(ii) σ-algebras are closed under countable intersections, while topologies are only required to be closed under finite intersections

Lemma

An algebra $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable disjoint unions, that is,

$$\left(\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \text{ disjoint} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}\right) \iff \mathcal{A} \text{ is a } \sigma\text{-algebra}$$

Proof: The $\Longleftarrow$ direction is clear from the definition of a σ-algebra.

For the $\Longrightarrow$ direction, let $\mathcal{A}$ be an algebra closed under countable disjoint unions. Then let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$. Define a set

$$F_i = E_i \setminus \cup_{j=1}^{i-1} E_j = E_i \cap \left(\cup_{j=1}^{i-1} E_j\right)^c = E_i \cap \left(\cap_{j=1}^{i-1} E_j^c\right) \in \mathcal{A}.$$

To see this a sequence of disjoint sets, suppose by contradiction $x \in F_a \cap F_b$ with $a < b$ without loss of generality. Then $x \in E_b$ and $x \notin \cup_{j=1}^{b-1} E_j$, so $x \notin E_a$, contradicting $x \in F_a$.

Since $E_i \subseteq F_i$, we have $\cup_{i=1}^{\infty} E_i \subseteq \cup_{i=1}^{\infty} F_i$. Now let $x \in \cup_{i=1}^{\infty} F_i$, so $x \in F_k$ for some $k \in \mathbb{N}$. Thus $x \in E_k$, so $x \in \cup_{i=1}^{\infty} E_i$.

Then $\{F_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of disjoint sets, so $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i \in \mathcal{A}$.

□

Exercise

Any algebra that is closed under countable increasing unions is a σ-algebra. (We say that an algebra $\mathcal{A}$ is closed under countable increasing unions if, for all $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ with $E_i \subseteq E_{i+1}$ for all i, $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$.)

Proof: Let $\mathcal{A}$ be an algebra closed under countable increasing unions. Then let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$. Define a set

$$F_i = \bigcup_{j=1}^{i} E_j \in \mathcal{A}$$

Notice that $F_i \subseteq F_{i+1}$, so $\{F_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ is a countable increasing sequence. Now $E_i \subseteq F_i$ is clear so $\cup_{i=1}^{\infty} E_i \subseteq \cup_{i=1}^{\infty} F_i$. Let $x \in \cup_{i=1}^{\infty} F_i$, so $x \in F_j$ for some $j \in \mathbb{N}$, meaning $x \in E_k$ for some $1 \leq k \leq j$. Thus $x \in \cup_{i=1}^{\infty} E_i$.

Then we have $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i \in \mathcal{A}$.

□

Note

Of our examples of algebras, notice that (i) and (ii) are σ-algebras, but (iii) and (iv) are not if $|X| = +\infty$.

Definition: Measurable Space and Measurable Set

Given a nonempty set X and a σ-algebra $\mathcal{M} \subseteq 2^X$, we call $(X, \mathcal{M})$ a **measurable space** and $E \in \mathcal{M}$ a **measurable set**.

Lemma

Given any nonempty collection $\mathcal{C}$ of σ-algebras on X, then $\cap\, \mathcal{C} = \{E \subseteq X : E \in \mathcal{A}, \forall \mathcal{A} \in \mathcal{C}\}$ is a σ-algebra.

Proof: Let $\mathcal{C}$ be a nonempty collection of σ-algebras on X. Take $\mathcal{A} \in \mathcal{C}$, which can be done since $\mathcal{C}$ is nonempty.

We first show that $\mathcal{C}$ is closed under countable unions. Let $\{E_i\}_{i=1}^{\infty} \subseteq \cap\, \mathcal{C}$. Then for each E_i, we have $E_i \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, we have $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$. Then since $\mathcal{A}$ was arbitrary, this is true for all $\mathcal{A} \in \mathcal{C}$. But this means $\cup_{i=1}^{\infty} E_i \in \cap\, \mathcal{C}$.

We now show that $\mathcal{C}$ is closed under complements. Let $E \in \cap\, \mathcal{C}$. Then $E \in \mathcal{A}$, so $E^c \in \mathcal{A}$. Then since $\mathcal{A} \in \mathcal{C}$ was arbitrary, $E^c \in \cap\, \mathcal{C}$.

□

Proposition

Given $E \subseteq 2^X$, there always exists a smallest σ-algebra containing E, which we denote by $\mathcal{M}(E)$ and refer to as the σ-algebra generated by E.

Note that by smallest σ-algebra, we mean that all σ-algebras $\mathcal{F}$ containing E satisfy $\mathcal{M}(E) \subseteq \mathcal{F}$.

Proof: Let $E \subseteq 2^X$. Let $\mathcal{C} = \{\mathcal{A} : \mathcal{A} \subseteq 2^X \text{ is a } \sigma\text{-algebra}, E \subseteq A\}$, i.e., this is the collection of σ-algebras containing E. Note since $2^X \in \mathcal{C}$, $\mathcal{C}$ is nonempty.

Now by our previous lemma, $\cap\, \mathcal{C}$ is a σ-algebra. By definition of $\mathcal{C}$, $E \subseteq \cap\, \mathcal{C}$ and $\forall \mathcal{A} \in \mathcal{C}, \cap\, \mathcal{C} \subseteq \mathcal{A}$.

□

With those basic properties of σ-algebras proven, we specify further, and investigate a particular very important class of σ-algebras.

Definition: The Borel σ-algebra

The **Borel σ-algebra** of X, denoted $B(X)$, is the σ-algebra generated by a topology τ. Its elements are called **Borel sets**.

At this point, with so many definitions introduced in quick succession, we might wonder what exactly a Borel σ-algebra looks like. To aid with this, we introduce some quick notation:

Notation

Given some $\mathcal{F} \subseteq 2^X$, denote:

$$\begin{aligned} \mathcal{F}^{\sigma} &:= \text{all countable unions of elements of } \mathcal{F} \\ \mathcal{F}^{\delta} &:= \text{all countable intersections of elements of } \mathcal{F} \\ \overline{\mathcal{F}} &:= \text{all countable complements of elements of } \mathcal{F} \end{aligned}$$

Then we can visualize Borel sets by building them from the "inside out", a process which gives us the **Borel hierarchy**:

$$\tau \to \tau^{\delta} \to \tau^{\delta} \cup \overline{\tau^{\delta}} \to \left(\tau^{\delta} \cup \overline{\tau^{\delta}}\right)^{\sigma} \to \underbrace{\cdots}_{\substack{\text{uncountably} \\ \text{many steps}}} \to B(X)$$

Lemma

Fix an open set $U \subseteq \mathbb{R}$. There exist $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^{\infty} (a_n, b_n)$.

Proof: Fix an open set $U \subseteq \mathbb{R}$. Since U is open, for each $x \in U$, there exists an open interval of x contained in U. So we have $x \in (a_i, b_i) \subseteq U$, and now by density of $\mathbb{Q}$ in $\mathbb{R}$, we can find rational numbers p_i and q_i such that $x \in (p_i, q_i) \subseteq U$. This implies we can write

$$U = \bigcup_{\substack{(p_i,q_i)\subseteq U \\ p,q\in\mathbb{Q}}} (p_i, q_i)$$

and this union is countable since $\mathbb{Q} \times \mathbb{Q}$ is countable. Thus, we can order the (p_i, q_i) pairs, for example by taking the p_i's in increasing order.

□

Proposition

The Borel σ-algebra of $\mathbb{R}$, denoted $B_{\mathbb{R}}$, is generated by

(i) Open intervals $\mathcal{E}_3 := \{(a, b) : a < b\}$
(ii) Half-open intervals $\mathcal{E}_5 := \{[a, b) : a < b\}$
(iii) Open rays $\mathcal{E}_7 := \{(a, +\infty) : a \in \mathbb{R}\}$

Proof:

(i) $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{B}_{\mathbb{R}}$: Let $(a, b] \in \mathcal{E}_3$. Notice that

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right),$$

where each $\left(a, b + \frac{1}{n}\right)$ is in the topology on $\mathbb{R}$, meaning that it is in $\mathcal{B}_{\mathbb{R}}$. Then $\mathcal{B}_{\mathbb{R}}$ is closed under countable intersections as a σ-algebra, so $(a, b] \in \mathcal{B}_{\mathbb{R}}$. So $\mathcal{E}_3 \subseteq \mathcal{B}_{\mathbb{R}}$, which clearly implies $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{B}_{\mathbb{R}}$.

$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_3)$: Let $U \subseteq \mathbb{R}$ be open. By Problem 3, there exist sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^{\infty} (a_n, b_n)$. Now observe

$$(a_i, b_i) = \bigcup_{n=1}^{\infty} \left(a_i, b_i - \frac{1}{n}\right].$$

for each a_i, b_i in the sequences. Thus

$$U = \bigcup_{i=1}^{\infty} (a_i, b_i) = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \left(a_i, b_i - \frac{1}{n}\right]$$

Now since each $\left(a_i, b_i - \frac{1}{n}\right] \in \mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_3)$, and σ-algebras are closed under countable unions, we have $U \in \mathcal{M}(\mathcal{E}_3)$. But now $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_3)$, since the Borel σ-algebra is generated by the open sets in $\mathbb{R}$.

(ii) $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{B}_{\mathbb{R}}$: Let $(a, +\infty) \in \mathcal{E}_5$. Notice that (a, n) is clearly open in the topology on $\mathbb{R}$, meaning that it is in $\mathcal{B}_{\mathbb{R}}$.

Then $\mathcal{B}_{\mathbb{R}}$ is closed under countable unions as a σ-algebra, so $(a, +\infty) \in \mathcal{B}_{\mathbb{R}}$. So $\mathcal{E}_5 \subseteq \mathcal{B}_{\mathbb{R}} \implies \mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{B}_{\mathbb{R}}$.

$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_5)$: Let $U \subseteq \mathbb{R}$ be open. By Problem 3, there exist sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^{\infty} (a_n, b_n)$. Now observe

$$(a_i, b_i) = (a_i, +\infty) \cap (-\infty, b_i) = (a_i, +\infty) \cap [b_i, +\infty)^c = (a_i, +\infty) \cap \left(\bigcap_{n=1}^{\infty} \left(b_i - \frac{1}{n}, +\infty\right)\right)^c.$$

for each a_i, b_i in the sequences. Thus

$$U = \bigcup_{i=1}^{\infty}(a_i, b_i) = \bigcup_{i=1}^{\infty}\left[(a_i, +\infty) \cap \left(\bigcap_{n=1}^{\infty}\left(b_i - \frac{1}{n}, +\infty\right)\right)^c\right]$$

Now since each $(a_i + \infty), \left(b_i - \frac{1}{n}, +\infty\right) \in \mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_5)$, and σ-algebras are closed under countable unions, countable intersections and complements, we have $U \in \mathcal{M}(\mathcal{E}_5)$. But now $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_5)$, since the Borel σ-algebra is generated by the open sets in $\mathbb{R}$.

(iii) $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{B}_\mathbb{R}$: Let $[a, +\infty) \in \mathcal{E}_7$. Notice that

$$[a, +\infty) = \bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, +\infty\right),$$

where each $\left(a + \frac{1}{n}, +\infty\right)$ is in the topology on $\mathbb{R}$, meaning that it's in $\mathcal{B}_\mathbb{R}$.

Then $\mathcal{B}_\mathbb{R}$ is closed under countable unions and intersections as a σ-algebra, so $[a, +\infty) \in \mathcal{B}_\mathbb{R}$. So $\mathcal{E}_7 \subseteq \mathcal{B}_\mathbb{R} \implies \mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{B}_\mathbb{R}$.

$\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_7)$: Let $U \subseteq \mathbb{R}$ be open. By Problem 3, there exist sequences $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^{\infty} (a_n, b_n)$. Now observe

$$\begin{aligned}(a_i, b_i) &= \bigcup_{n=1}^{\infty}\left[a_i + \frac{1}{n}, b_i\right) \\ &= \bigcup_{n=1}^{\infty}\left[\left[a_i + \frac{1}{n}, +\infty\right) \cap (-\infty, b_i)\right] \\ &= \bigcup_{n=1}^{\infty}\left[\left[a_i + \frac{1}{n}, +\infty\right) \cap [b_i, +\infty)^c\right]\end{aligned}$$

for each a_i, b_i in the sequences. Thus

$$U = \bigcup_{i=1}^{\infty}(a_i, b_i) = \bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\left[\left[a_i + \frac{1}{n}, +\infty\right) \cap [b_i, +\infty)^c\right].$$

Now since each $\left[a_i + \frac{1}{n}, +\infty\right), [b_i, +\infty) \in \mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_7)$, and σ-algebras are closed under countable unions and complements, we have $U \in \mathcal{M}(\mathcal{E}_7)$. But now $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_7)$, since the Borel σ-algebra is generated by the open sets in $\mathbb{R}$.

□

Importantly, the Borel σ-algebra of $\mathbb{R}$ won't include Vitali sets, or any other pathological examples of sets that don't work with our desired notion of measure. That makes it a great candidate to use as a σ-algebra for our desired function.

Now, we can define the general notion of a measure, where we restrict the domain to a σ-algebra rather than the entire powerset.

2.C Measures

Definition: Measure and Measure Space

Given a measurable space $(X, \mathcal{M})$, a **measure** is a function $\mu : \mathcal{M} \to [0, +\infty]$ such that:

(i) $\mu(\emptyset) = 0$

(ii) Given a sequence of disjoint sets $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Recall the second property is called **countable additivity**. We call $(X, \mathcal{M}, \mu)$ a **measure space**.

Notice that translation invariance is **not** included. Only some measures have this property.

Example: Dirac Mass / Dirac Measure

Take the measurable space $(X, \mathcal{M}) = (X, 2^X)$. For $x_0 \in X$, define

$$\mu(A) = \begin{cases} 1 \text{ if } x_0 \in A \\ 0 \text{ otherwise}' \end{cases}$$

which is often denoted $\mu = \delta_{x_0}$.

Example: Counting Measure

Again take the measurable space $(X, \mathcal{M}) = (X, 2^X)$. Define $\mu(A) = |A| := \#$ of elements in A.

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Theorem: Measure Properties

For any measure space $(X, \mathcal{M}, \mu)$ and $A, B \in \mathcal{M}$ with $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$,

(i) $A \subseteq B \implies \mu(A) \leq \mu(B)$

(ii) $A \subseteq B$ and $\mu(A) < +\infty \implies \mu(B \setminus A) = \mu(B) - \mu(A)$

(iii) $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{n} \mu(A_i)$

(iv) $A_i \subseteq A_{i+1} \forall i \in \mathbb{N} \implies \mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to+\infty} \mu(A_i)$ **(continuity from below)**

(v) $A_i \supseteq A_{i+1} \forall i \in \mathbb{N}$ and $\mu(A_1) < +\infty \implies \mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to+\infty} \mu(A_i)$ **(continuity from above)**

Note that we can understand continuity from below intuitively by thinking of the sets as increasing upwards, like an inverted pyramid. Similarly, continuity from above can be thought of as starting at the top of an inverted pyramid and going down.

Proof:

(i) Shown in 2.B.

(ii) Let $A \subseteq B$. Then $\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$. Now since $\mu(A) < +\infty$, we have $\mu(B) - \mu(A) = \mu(B \setminus A)$.

(iii) Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus \cup_{i=1}^{n-1} A_i$. Then $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ are disjoint and $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$, with $B_i \subseteq A_i \forall i \in \mathbb{N}$. Now observe

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

where we invoked property (i) in the last step.

(iv) Suppose $A_i \subseteq A_{i+1} \forall i \in \mathbb{N}$. Define $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i > 1$. By definition, $\cup_{i=1}^{n} B_i = A_n$. Also, $\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i$. Thus,

$$\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i).$$

But taking $n \to +\infty$ gives

$$\lim_{n\to+\infty} \mu(A_n) = \sum_{i=1}^{\infty} \mu(B_i) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

(v) Suppose $A_i \supseteq A_{i+1} \forall i \in \mathbb{N}$. Define $B_i = A_1 \setminus A_i$. By construction, $B_i \subseteq B_{i+1} \forall i \in \mathbb{N}$. Then

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty}(A_1 \setminus A_i) = \bigcup_{i=1}^{\infty} A_1 \cap A_i^c = A_1 \cap \left(\bigcup_{i=1}^{\infty} A_i^c\right) = A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i\right)^c = A_1 \setminus \bigcap_{i=1}^{\infty} A_i.$$

Thus,

$$\begin{aligned}
\mu(A_1) &= \mu\left(A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i\right)\right) + \mu\left(A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right)\right) \\
&= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \lim_{i\to\infty} \mu(B_i) && \text{by (iv)} \\
&= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \lim_{i\to\infty} \mu(A_1 \setminus A_i) \\
&= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \lim_{i\to\infty} \mu(A_1) - \mu(A_i). && \text{by (ii)}
\end{aligned}$$

Now using our assumption that $\mu(A_1) < +\infty$, we can subtract it from both sides, giving the result.

□

Example: Why $\mu(A_1) < +\infty$ is required

Take $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$ and $\mu(E) = |E|$. Now let our sets be $A_i = \{n \in \mathbb{N} : n \geq i\}$.

Then we get $\cap_{i=1}^{\infty} A_i = \emptyset$ and $\mu(\cap_{i=1}^{\infty} A_i) = 0 \neq +\infty = \lim_{i\to+\infty} \mu(A_i)$.

We proceed with some important terminology that characterizes measures and the sets they act on:

Definition: Finite Measure

- We call μ a **finite measure** if $\mu(X) < +\infty$
- Further, μ is a **σ-finite measure** if $\exists\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ such that $\cup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty \forall i \in \mathbb{N}$

Definition: Null Set, μ-almost everywhere

$E \subseteq X$ is a **null set** of μ if $E \in \mathcal{M}$ and $\mu(E) = 0$

We say a property holds **μ-almost everywhere** if the set of points where it fails is a null set

Example: μ-almost everywhere

Suppose we have some $f : X \to \mathbb{R}$. Then $f = 0$ almost everywhere if $\{x : f(x) \neq 0\} \in \mathcal{M}$ and $\mu(\{x : f(x) \neq 0\}) = 0$, (that is, if $\{f \neq 0\}$ is a null set).

2.D Limit Inferior and Limit Superior for Sets

Definition: Limit Inferior and Limit Superior for Sets

Given a collection of sets $\{E_i\}_{i=1}^{\infty}$, define

$$\limsup_{i\to+\infty} E_i := \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i, \qquad \liminf_{i\to+\infty} E_i := \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} E_i.$$

Proposition: Characterization of* lim sup *and* lim inf *on Sets

We can characterize the lim sup and lim inf intuitively by the following:

$$\liminf_{i\to+\infty} E_i := \{x : x \in E_i \text{ for all but finitely many } i\}$$
$$\limsup_{i\to+\infty} E_i := \{x : x \in E_i \text{ for infinitely many } i\}.$$

Proof: Notice

$$\begin{aligned} x \in \limsup_{i\to+\infty} E_i &\iff x \in \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i \\ &\iff x \in \cup_{i=k}^{\infty} E_i \forall k \geq 1 \\ &\iff x \in E_j \text{ for some } j \geq k \forall k \geq 1 \\ &\iff x \in E_i \text{ for infinitely many } i \end{aligned}$$

and

$$\begin{aligned} x \in \liminf_{i\to+\infty} &\iff x \in \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} E_i \\ &\iff \exists k \in \mathbb{N} \text{ such that } x \in \cap_{i=k}^{\infty} E_i \\ &\iff \exists k \in \mathbb{N} \text{ such that } x \in E_i \forall i \geq k \\ &\iff x \in E_i \text{ for all but finitely many } i. \end{aligned}$$

□

Exercise

Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < +\infty$. Suppose $A_1, A_2, \ldots$ are sets in $\mathcal{M}$ with $\mu(A_i) \geq c > 0$ for all i. Let Z be the set of elements $x \in X$ that belong to infinitely many of the A_i's. Prove that $\mu(Z) \geq c$.

Proof: From the previous proposition, we can write $Z = \limsup_{i \to +\infty} A_i$. Now define $B_k = \cup_{i=k}^{\infty} A_i$ so that $Z = \cap_{k=1}^{\infty} B_k$. Observe that $B_{k+1} \subseteq B_k$, so by continuity from above (which is justified since $\mu(X) < +\infty$), we have

$$\mu(Z) = \mu(\cap_{k=1}^{\infty} B_k) = \lim_{k \to +\infty} \mu(B_k).$$

Now since $A_k \subseteq B_k$, by monotonicity we have

$$\mu(B_k) \geq \mu(A_k) \geq c.$$

Now take $k \to +\infty$. Thus we have $\mu(Z) \geq c$ as desired.

□

3 Outer Measures

3.A Outer Measures

Now, we have a good definition for a measure, since we restricted to thinking about the individual "building blocks" rather than every possible subset of X. We also introduced some basic measures.

The key now is to show this notion solves our original difficulty: finding a measure that properly assigns lengths to closed intervals $[a, b]$, and find an associated σ-algebra. In fact, we'll do this in a general way: given a notion of size, how can we find an associated σ-algebra and measure?

Our general plan will be as follows:

(i) Start with a collection of sets containing all sets we want to know how to measure (like 2^X)
(ii) Define a way to approximate the measure from the outside (called the **outer measure**)
(iii) Construct a σ-algebra using the outer measure (this can be done **Carathéodory's criterion**, shown later on)
(iv) Obtain an actual measure on that σ-algebra

Assuming our initial collection of sets is 2^X, we proceed with step 2.

Definition: Outer Measure

An **outer measure** on X is a function $\mu^* : 2^X \to [0, +\infty]$ such that

(i) $\mu^*(\emptyset) = 0$
(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$ (monotonicity)
(iii) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ (countable subadditivity)

The idea behind property (iii) is that we get a covering of the sets we want, even if there is some wasteful overlap. Then, we can make the bound tight and get equality, producing a measure. Also, property (ii) is needed because while it follows from countable additivity, it does not follow from countable subadditivity.

Remark

From (ii) and (iii), we have that $E \subseteq \cup_{i=1}^{\infty} A_i$ then $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

In fact, showing this plus (i) is enough to show a function is an outer measure.

Proposition

If $\mu^* : 2^X \to [0, +\infty]$ satisfies

(i) $\mu^*(\emptyset) = 0$ and
(ii) $E \subseteq \cup_{i=1}^{\infty} A_i \Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$,

then μ^* is an outer measure.

Now comes the outer measure for the measure we have been trying to construct:

Example: Lebesgue Outer Measure

Define $\mu^* : 2^{\mathbb{R}} \to [0, +\infty]$ by $\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \cup_{i=1}^{\infty} (a_i, b_i]\right\}$ with $a_i \leq b_i$.

We will show that μ^* has many of the properties we wanted:

- μ^* is an outer measure
- μ^* is translation invariant
- $\mu^*((a,b]) = b - a \forall a \leq b$

and it will become a measure when it is restricted to the relevant σ-algebra. Again, how do we actually do this? The answer is Caratheodory's Theorem, which we will now build up to.

3.B Caratheodory's Theorem

Definition: μ^* measurable

Given an outer measure μ^* on X, $A \subseteq X$ is **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \subseteq X$.

We can read this as "A is measurable we can break any set E apart nicely." We use the notation $\mathcal{M}_{\mu^*} := \{A \subseteq X : A \text{ is } \mu^* \text{ measurable}\}$. This is also called **Caratheodory's criterion**.

Remark

Suppose we want to show $A \in \mathcal{M}_{\mu^*}$ By countable subadditivity, we always have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

so all we need to do is show the $\geq$ direction.

Note

$\mathcal{M}_{\mu^*}$ is not the "largest" σ-algebra on which μ^* is a measure. (Update with details later.)

Proposition

For any outer measure μ^*, if $\mu^*(B) = 0$, then $B \in \mathcal{M}_{\mu^*}$.

Proof: For any $E \subseteq X$, by monotonicity we have

$$\begin{aligned} \mu^*(E) &\geq 0 + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \end{aligned}$$

Thus, $B \in \mathcal{M}_{\mu^*}$.

□

Proposition

(i) $\mathcal{M}_{\mu^*}$ is an algebra
(ii) Given $\{B_i\}_{i=1}^n \subseteq \mathcal{M}_{\mu^*}$ disjoint, we have $\mu^*(E \cap (\cup_{i=1}^n B_i)) = \sum_{i=1}^n \mu^*(E \cap B_i) \forall E \subseteq X$
(iii) μ^* is finitely additive

Proof:

(i) Since $\mu^*(\emptyset) = 0$, by the previous proposition we have $\emptyset \in \mathcal{M}_{\mu^*}$, so $\mathcal{M}_{\mu^*}$ is nonempty. Further, $\mathcal{M}_{\mu^*}$ is closed under complements, since we can just replace E with E^c in the definition.

To see that $\mathcal{M}_{\mu^*}$ is closed under finite unions, it suffices to show that $A, B \in \mathcal{M}_{\mu^*} \Rightarrow A \cup B \in \mathcal{M}_{\mu^*}$. Suppose $A, B \in \mathcal{M}_{\mu^*}$. Fix $E \subseteq X$.

Then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap (A \cup B)^c) \\ &= \mu^*(E \cap (A \cup A^c \cup B)) + \mu^*(E \cap (A \cup B)^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).\end{aligned}$$

(ii) Suppose $\{B_i\}_{i=1}^n \subseteq \mathcal{M}_{\mu^*}$ disjoint. Fix $E \subseteq X$. Proceed by induction. The base case $n = 1$ is trivial, and now suppose it holds for $n - 1$. Then

$$\begin{aligned}\mu^*(E \cap (\cup_{i=1}^n B_i)) &= \mu^*(E \cap (\cup_{i=1}^n B_i) \cap B_n) + \mu^*(E \cap (\cup_{i=1}^n B_i) \cap B_n^c) \\ &= \mu^*(E \cap B_n) + \mu^*(E \cap ((\cup_{i=1}^{n-1} B_i \cap B_n^c) \cup (B_n \cap B_n^c))) \\ &= \mu^*(E \cap B_n) + \mu^*(E \cap (\cup_{i=1}^{n-1} B_i)) \\ &= \sum_{i=1}^n \mu^*(E \cap B_i).\end{aligned}$$

(iii) Take $E = B_i$ in (ii).

□

Proposition

Given $\{B_i\}_{i=1}^\infty \subseteq \mathcal{M}_{\mu^*}$ disjoint,

$$\mu^*(E) = \sum_{i=1}^\infty \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c)$$

for all $E \subseteq X$.

Proof: ($\leq$) By subadditivity we have

$$\begin{aligned}\mu^*(E) &\leq \mu^*(E \cap B_i) + \mu^*(E \cap B_i^c) \\ &\leq \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c) \\ &\leq \sum_{i=1}^\infty \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c).\end{aligned}$$

($\geq$) By (i) of the previous proposition, $\mathcal{M}_{\mu^*}$ is closed under finite unions, so for any $n \in \mathbb{N}$ we have

$$\bigcup_{i=1}^n B_i \in \mathcal{M}_{\mu^*}.$$

Thus we have

$$\mu^*(E) = \mu^*(E \cap (\cup_{i=1}^n B_i)) + \mu^*(E \cap (\cup_{i=1}^n B_i)^c)$$
$$= \sum_{i=1}^n \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^n B_i)^c)$$

where we used part (ii) of the previous proposition. Now take $n \to +\infty$ and we get the result.

□

Lecture 4 — Oct 7

Theorem: Caratheodory's Theorem

Given an outer measure μ^*,

(i) $\mathcal{M}_{\mu^*}$ is a σ-algebra

(ii) μ^* is a measure on $\mathcal{M}_{\mu^*}$

Proof:

(i) Note that we proved $\mathcal{M}_{\mu^*}$ is an algebra two propositions ago, so we need only show that it is closed under countable unions. In fact, from this lemma, it is sufficient to show closure under countable disjoint unions.

Thus suppose $\{B_i\}_{i=1}^\infty \subseteq \mathcal{M}_{\mu^*}$ disjoint. Fix $E \subseteq X$. Then from the previous proposition we have

$$\mu^*(E) = \sum_{i=1}^\infty \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c)$$
$$\geq \mu^*(E \cap (\cup_{i=1}^\infty B_i)) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c)$$

where the second line follows from subadditivity.

(ii) Since μ^* is an outer measure, we already know $\mu(\emptyset) = 0$. To show countable additivity, we can take $E = \cup_{i=1}^\infty B_i$ in the previous proposition.

□

Remark

Although Caratheodory's Thoerem tells us that any outer measure μ^* gives us a measure when we restrict it to the collection of μ^* measurable sets, it turns out that this is not, in general, the largest σ-algebra on which μ^* becomes a measure. The following exercise shows this.

Exercise

Consider the set $X = \{1, 2, 3\}$. Define an outer measure as follows:

$$\mu^*(A) = \begin{cases} 0 & \text{if } |A| = 0 \\ 1 & \text{if } |A| = 1, 2 \\ 2 & \text{if } |A| = 3. \end{cases}$$

(a) Prove that μ^* is an outer measure on X.

(b) Prove that the collection of μ^* measurable sets is $\{\emptyset, X\}$.

(c) Prove that $\mathcal{A} := \{\emptyset, \{1\}, \{2, 3\}, X\}$ is a σ-algebra.

(d) Prove that $\mu^* \mid_{\mathcal{A}}$ is a measure.

This shows that the Caratheodory σ-algebra $\mathcal{M}_{\mu^*}$ is not, in general, the largest σ-algebra on which σ^* can be restricted to be a measure.

Proof:

(i) We confirm each property of an outer measure:

- $\mu^*(\emptyset) = 0$ since $|\emptyset| = 0$
- Let $A \subseteq B$. Observe that $\mu^*(A)$ is clearly nondecreasing as a function of $|A|$, meaning that
$$A \subseteq B \Rightarrow |A| \leq |B| \Rightarrow \mu(A) \leq \mu(B).$$
- Let $\{A_i\}_{i=1}^{\infty} \subseteq X$. Let $A := \cup_{i=1}^{\infty} A_i$. Clearly we have $\mu^*(A) \leq |A|$.
  - If $|A| = 0$, we have $\mu^*(A) = 0 \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, just from nonnegativity of the outer measure.
  - Suppose $|A| \in \{1, 2\}$. Notice that if all $A_i = \emptyset$ then we would have $A = \emptyset$, a contradiction. So at least one A_i is nonempty, meaning $\exists j$ such that $\mu^*(A_j) \geq 1$. Thus $\mu^*(A) = 1 \leq \mu^*(A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.
  - Suppose $|A| = 3$, in which case $\mu^*(A) = 2$. Then the A_i's must collectively cover each of $\{1, 2, 3\}$.
    - If any A_i has $|A_i| = 3$, then $\sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*(A_i) \geq 2$ and we're done.
    - If any A_i has $|A_i| = 2$, then there must be another set A_j covering the last element, with $|A_j| \geq 1$. Thus we would have $\sum_{n=1}^{\infty} \mu^*(A_n) \geq \mu^*(A_i) + \mu^*(A_j) \geq 1 + 1 = 2$, and we're done.
    - If all nonempty A_i have $|A_i| = 1$, we need at least three of them to cover A. Thus there exist A_i, A_j, A_k, each with outer measure 1 that cover A, meaning that $\sum_{i=1}^{\infty} \mu^*(A_i) \geq 3$ and we're done.

(ii) ($\supseteq$) Notice that $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \cap X)$, so $\{\emptyset, X\}$ are both trivially μ^* measurable.

($\subseteq$) Let $A \subseteq X$ be a μ^* measurable set.

- First suppose that $|A| = 1$. Without loss of generality take $A = \{1\}$, and consider $E = \{1, 2\}$. Then $1 = \mu^*(\{1, 2\}) \neq \mu^*(\{1\}) + \mu^*(\{2\}) = 2$, so this doesn't work.
- Now suppose that $|A| = 2$. Without loss of generality take $A = \{1, 2\}$, and consider $E = \{2, 3\}$. Then $1 = \mu^*(\{2, 3\}) \neq \mu^*(\{2\}) + \mu^*(\{3\}) = 2$, so this also doesn't work.

Thus we must have that $|A| \in \{0, 3\}$, showing that $A \subseteq \{\emptyset, X\}$.

(iii) First notice that $\emptyset^c = X$ and $\{1\}^c = \{2, 3\}$, showing closure under complement. Next notice that the only nontrivial union is $\{1\} \cup \{2, 3\} = X$, showing closure under countable unions.

(iv) Notice that $\emptyset \in \mathcal{A}$ so $\mu^* \mid_{\mathcal{A}} (\emptyset) = 0$. For the trivial disjoint sets, notice
$$\mu^* \mid_{\mathcal{A}} (\emptyset \cup \emptyset) = \mu^* \mid_{\mathcal{A}} (\emptyset) = 0 = 0 + 0 = \mu^* \mid_{\mathcal{A}} (\emptyset) + \mu^* \mid_{\mathcal{A}} (\emptyset)$$
$$\mu^* \mid_{\mathcal{A}} (\emptyset \cup A) = \mu^* \mid_{\mathcal{A}} (A) = \mu^* \mid_{\mathcal{A}} (A) + 0 = \mu^* \mid_{\mathcal{A}} (A) + \mu^* \mid_{\mathcal{A}} (\emptyset)$$

The only nontrivial disjoint sets in $\mathcal{A}$ are $\{1\}$ and $\{2, 3\}$, and we have
$$\mu^* \mid_{\mathcal{A}} (\{1\} \cup \{2, 3\}) = \mu^* \mid_{\mathcal{A}} (X) = 2 = \mu^* \mid_{\mathcal{A}} (\{1\}) + \mu^* \mid_{\mathcal{A}} (\{2, 3\})$$

which shows that $\mu^* \mid_{\mathcal{A}}$ is indeed closed under disjoint unions, so it is indeed a measure.

□

Regardless, now we can take μ^* in Caratheodory's Theorem to be the Lebesgue outer measure, as discussed previously. This gives us the **Lebesgue measure**, which has our desired properties.

Instead of proving that those properties hold directly though, it turns out that they hold for a larger class of outer measures, and the properties of the Lebesgue measure will become apparent as a special case. Thus, we generalize first.

3.C Lebesgue-Stieltjes Outer Measure

We can generalize the Lebesgue outer measure to the **Lebesgue-Stieltjes Outer Measure**, which concerns itself with nondecreasing, right continuous functions. We briefly review the definition of right continuity:

Definition: Right continuous

A function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is **right continuous** if $\forall x \in \mathbb{R}$ we have $\lim_{y \to x^+} F(y) = \lim_{y \searrow x} F(y) = F(x)$.

Definition: Lebesgue-Stieltjes Outer Measure

Given $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous, define the **Lebesgue-Stieltjes Outer Measure** $\mu_F^* : 2^{\mathbb{R}} \to [0, +\infty]$ by

$$\mu_F^*(A) = \inf\left\{\sum_{i=1}^{\infty} |F(b_i) - F(a_i)|\right\} = \inf\left\{\sum_{i=1}^{\infty} |I_i|_F\right\}$$

where $A \subseteq \cup_{i=1}^{\infty} (a_i, b_i]$ and $a_i \leq b_i$.

Note that the Lebesgue Outer Measure from before is the case where $F(x) = x$.

Theorem

For any $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous, μ_F^* is an outer measure on $\mathbb{R}$.

Proof: By definition, $\mu_F^* \geq 0$. Since $\emptyset \subseteq \cup_{i=1}^{\infty} (0, 0]$, $\mu_F^*(\emptyset) \leq \sum_{i=1}^{\infty} 0 = 0$, we have $\mu_F^*(\emptyset) \leq 0$, so $\mu_F^*(\emptyset) = 0$.

It remains to show $A \subseteq \cup_{i=1}^{\infty} B_i \Rightarrow \mu_F^*(A) \leq \sum_{i=1}^{\infty} \mu_F^*(B_i)$ (which sufficient to prove μ_F^* is an outer measure via this proposition).

Now fix $A \subseteq \cup_{i=1}^{\infty} B_i$. Without loss of generality, $\sum_{i=1}^{\infty} \mu_F^*(B_i) < +\infty$ (because otherwise this is trivially true). Thus $\mu_F^*(B_i) < +\infty \forall i \in \mathbb{N}$.

Then let $\varepsilon > 0$ be fixed. Since the outer measure of each B_i is finite, for each B_i there exists $\left\{I_i^{j,\varepsilon}\right\}_{j=1}^{\infty}$ such that $B_i \subseteq \cup_{j=1}^{\infty} I_i^{j,\varepsilon}$. Then we have (using the characterization of the infimum in $\overline{\mathbb{R}}$),

$$\mu_F^*(B_i) \leq \sum_{j=1}^{\infty} \left|I_i^{j,\varepsilon}\right|_F \leq \mu_F^*(B_i) + \frac{\varepsilon}{2^i}.$$

Furthermore,

$$A \subseteq \cup_{i,j=1}^{\infty} \left|I_i^{j,\varepsilon}\right|_F.$$

Thus

$$\mu^*(A) \leq \sum_{i,j=1}^{\infty} \left|I_i^{j,\varepsilon}\right|_F \leq \left(\sum_{i=1}^{\infty} \mu_F^*(B_i)\right) + \varepsilon$$

Now taking $\varepsilon \to 0$ gives the result.

□

Theorem

For $F : \mathbb{R} \to \mathbb{R}$ nondecreasing, right continuous and all $a, b \in \mathbb{R}$ with $a \leq b$, we have

$$\mu_F^*((a,b]) = F(b) - F(a).$$

Proof: ($\leq$): Notice that $(a,b] \subseteq (a,b] \cup \emptyset \cup \emptyset \cup \cdots$ is a valid covering, immediately giving us $\mu_F^*((a,b]) \leq F(b) - F(a)$.

($\geq$): Without loss of generality let $a < b$. Suppose $(a,b] \subseteq \cup_{i=1}^{\infty} (a_i, b_i]$ with $a_i \leq b_i$. Fix $\varepsilon > 0$. Since F is right continuous, $\exists \delta_i > 0$ such that

$$b_i \leq x \leq b_i + \delta_i \Rightarrow |F(x) - F(b_i)| < \frac{\varepsilon}{2^i}.$$

In particular, take $x = b_i + \delta_i$ and observe $F(b_i + \delta_i) \geq F(b_i)$ since it is nondecreasing,

$$0 \leq F(b_i + \delta_i) - F(b_i) < \frac{\varepsilon}{2^i}.$$

Further,

$$(a,b] \subseteq \cup_{i=1}^{\infty} (a_i, b_i] \subseteq \cup_{i=1}^{\infty} (a_i, b_i + \delta_i).$$

Thus we have an open cover for $(a,b]$, and since $[a+\varepsilon, b]$ is compact, there exists a finite subcover, meaning that after reindexing we have

$$[a+\varepsilon, b] \subseteq \cup_{i=1}^{N} (a_i, b_i + \delta_i).$$

Thus up to reindexing again we have $b_1 + \delta_1 < \cdots < b_N + \delta_N$.

Further, for each $i = 1, \ldots, N$ we have $b_i + \delta_i \in (a_{i+1}, b_{i+1} + \delta_{i+1})$, since we would have a gap in the cover otherwise. Now since F is nonincreasing, we have

$$\begin{aligned}
F(b) - F(a+\varepsilon) &\leq F(b_N + \delta_N) - F(a_1) \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(a_{i+1}) - F(a_i) \\
&\leq F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(b_i + \delta_i) - F(a_i) \\
&= \sum_{i=1}^{N} F(b_i + \delta_i) - F(a_i) \\
&\leq \sum_{i=1}^{N} F(b_i) - F(a_i) + \frac{\varepsilon}{2^i} \\
&\leq \sum_{i=1}^{\infty} F(b_i) - F(a_i) + \frac{\varepsilon}{2^i}
\end{aligned}$$

$$= \varepsilon + \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

Now sending $\varepsilon \to 0$ gives the result, noting again that F is nondecreasing.

□

Lecture 5 Oct 9

Definition: Lebesgue Measure

In the previous two theorems, taking $F(x) = x$ gives us that
(i) μ_* is an outer measure
(ii) $\mu_*((a, b]) = b - a$

And from Caratheodory's Theorem, (i) gives us that μ^* is a measure on the σ-algebra $\mathcal{M}_{\mu^*}$. We call this measure the **Lebesgue measure**; it's what we were originally after. Since it's a measure, we can see it satifies countable additivity, and by definition, it gives the "right length" to intervals. It remains to show the translation invariance property, which we will do shortly.

Notation: Lebesgue Measure

When $F(x) = x$, write

$$\begin{aligned} \lambda^* &:= \mu_F^* && \text{(the Lebesgue outer measure)} \\ \mathcal{M}_{\lambda^*} &:= \mathcal{M}_{\mu_F^*} && \text{(the Lebesgue measurable sets)} \\ \lambda &:= \lambda^* \mid_{\mathcal{M}_{\lambda^*}} && \text{(the Lebesgue measure)} \end{aligned}$$

Definition: Lebesgue-Stieltjes Measure

Generalizing the above, we now know we can take any $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous and produce a measure:

$$\mu_F := \mu_F^* \mid_{\mathcal{M}_{\mu_F^*}}.$$

We call this the **Lebesgue-Stieltjes Measure**.

Theorem

For any nondecreasing, right continuous function $F : \mathbb{R} \to \mathbb{R}$, we have $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F^*}$.

Proof: Since $(-\infty, b]$ generates the Borel σ-algebra, it suffices to show that $(-\infty, b] \in \mathcal{M}_{\mu_F^*}\ \forall b \in \mathbb{R}$, that is, we must show that $\forall E \subseteq \mathbb{R}, b \in \mathbb{R}$, we have

$$\mu_F^*(E) \geq \mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c).$$

Take $E \subseteq \mathbb{R}$. Without loss of generality, $\mu_F^*(E) < +\infty$. Fix $\varepsilon > 0$. By definition, $\exists \{(a_i, b_i]\}_{i=1}^{\infty}$ such that $E \subseteq \cup_{i=1}^{\infty} (a_i, b_i]$ and $\sum_{i=1}^{\infty} F(b_i) - F(a_i) < \mu_F^*(E) + \varepsilon$. Observe

$$(a_i, b_i] \cap (-\infty, b] \subseteq (a_i, \min\{b_i, b\})$$

$$(a_i, b_i] \cap (b, +\infty) \subseteq (\max\{a_i, b\}, b_i)$$

so

$$E \cap (-\infty, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, \min\{b_i, b\})$$
$$E \cap (b, +\infty) \subseteq \bigcup_{i=1}^{\infty} (\max\{a_i, b\}, b_i).$$

Thus

$$\mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c) \leq \sum_{i=1}^{\infty} \underbrace{F(\min\{b_i, b\}) - F(a_i)}_{(\mathrm{I})_i} + \sum_{i=1}^{\infty} \underbrace{F(b_i) - F(\max\{a_i, b\})}_{(\mathrm{II})_i}.$$

Consider the ith term of the sum:

(i) If $b \leq a_i$, then $(\mathrm{I})_i = 0$ and $(\mathrm{II})_i = F(b_i) - F(a_i)$
(ii) If $b > b_i$, then $(\mathrm{I})_i = F(b_i) - F(a_i)$ and $(\mathrm{II})_i = 0$
(iii) If $b \in (a_i, b_i]$, then $(\mathrm{I}) + (\mathrm{II}) = F(b) - F(a_i) + F(b_i) - F(b) = F(b_i) - F(a_i)$

So in general we have

$$\begin{aligned} \mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c) &\leq \sum_{i=1}^{\infty} F(b_i) - F(a_i) \\ &\leq \mu_F^*(E) + \varepsilon. \end{aligned}$$

Now taking $\varepsilon \to 0$ gives the result.

□

Theorem: Translation Invariance of the Lebesgue Measure

The Lebesgue Outer Measure λ^* is translation invariant on $2^{\mathbb{R}}$, and λ is translation invariant on $\mathcal{M}_{\lambda^*}$.

Proof: For any $a \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we have $A \subseteq \cup_{i=1}^{\infty} (a_i, b_i] \Leftrightarrow A + a \subseteq \cup_{i=1}^{\infty} (a_i + a, b_i + a]$, and both have the same "length" for $F(x) = x$. Thus $\lambda^*(A) = \lambda^*(A + a)$, so we have established translation invariance on $2^{\mathbb{R}}$.

Now we show translation invariance when we restrict to $\mathcal{M}_{\lambda}^*$. Suppose $A \in \mathcal{M}_{\lambda^*}$. Fix $a \in \mathbb{R}$. We want to show $A + a \in \mathcal{M}_{\lambda^*}$.

Fix $E \subseteq \mathbb{R}$. Note that for any $S \subseteq \mathbb{R}$, we have $(E - a) \cap S = [E \cap (S + a)] - a$ and $(S + a)^c = S^c + a$. Since $A \in \mathcal{M}_{\lambda^*}$,

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E - a) \\ &\geq \lambda^*((E - a) \cap A) + \lambda^*((E - a) \cap A^c) \\ &= \lambda^*([E \cap (A + a)] - a) + \lambda^*([E \cap (A^c + a)] - a) \\ &= \lambda^*(E \cap (A + a)) + \lambda^*(E \cap (A^c + a)) \\ &= \lambda^*(E \cap (A + a)) + \lambda^*(E \cap (A + a)^c). \end{aligned}$$

□

Thus, we have shown at last that the Lebesgue Measure solves our initial question.

Theorem: Characterization of the Cumulative Distribution Function

Suppose μ is a finite Borel measure on $\mathbb{R}$. Define $F:\mathbb{R}\to\mathbb{R}$ by $F(x):=\mu([-\infty,x])$ (this is called a **Cumulative Distribution Function**. Then

(i) F is nondecreasing and right continuous

(ii) $\mu=\mu_F$

Proof:

(i) To see that it is nondecreasing, notice $x\le y\Rightarrow F(x)=\mu((-\infty,x])\le\mu((-\infty,y])=F(y)$. To see that F is right continuous, notice that for any sequence $\{x_n\}_{n=1}^{\infty}\searrow x$, we have $\lim_{n\to+\infty}F(x_n)=\lim_{n\to+\infty}\mu((-\infty,x_n])=\lim_{n\to+\infty}\mu(\cap_{n=1}^{\infty}(-\infty,x_n])=\mu((-\infty,x_n])$, where we used continuity from above

(ii) Fix $a\le b$. Then

$$\mu((a,b])=\mu((-\infty,b]\setminus(-\infty,a])=\mu((-\infty,b])-\mu((-\infty,a])=F(b)-F(a)=\mu_F((a,b]).$$

($\le$) Now fix $E\in\mathcal{B}_{\mathbb{R}}$ and consider $\{(a_i,b_i]\}_{i=1}^{\infty}$ with $a_i\le b_i$ and $E\subseteq\cup_{i=1}^{\infty}(a_i,b_i]$. By subadditivity and monotonicity, we have

$$\mu(E)\le\mu(\cup_{i=1}^{\infty}(a_i,b_i])\le\sum_{i=1}^{\infty}\mu((a_i,b_i])\le\sum_{i=1}^{\infty}F(b_i)-F(a_i).$$

Now taking the infimum over all covers on the right hand side, we see $\mu(E)\le\mu_F^*(E)=\mu_F(E)$.

($\ge$) Observe that $\mu(\mathbb{R})=\lim_{n\to\infty}\mu((-n,n])=\lim_{n\to\infty}\mu_F((-n,n])=\mu_F(\mathbb{R})$, where we used continuity from below. Thus we can say

$$\mu(E)=\mu(\mathbb{R})-\mu(E^c)\ge\mu_F(\mathbb{R})-\mu_F(E^c)=\mu_F(E),$$

where we used the argument we made in the ($\le$) direction.

$\square$

Remark: Optimization Terminology

Given a nonempty set X, and a function $f:X\to\mathbb{R}\cup\{+\infty\}$, with $C\subseteq X$ and $C\ne\emptyset$, consider the optimization problem

$$M:=\inf_{x\in C}f(x)=\inf\{f(x):x\in C\}.$$

We call

- C the **constraint set**
- f the **objective function**
- M the **optimum**

By definition of the infimum, there exists a minimizing sequence x_n, that is, a sequence $\{x_n\}_{n=1}^{\infty}\subseteq C$ such that $\lim_{n\to+\infty}f(x_n)=M$.

If $\exists x\in C$ such that $f(x)=M$, then we call x an **optimizer** (since it obtains the optimum).

Lecture 6 Oct 14

Proposition

$$\mu_F(E) = \inf\left\{\sum_{i=1}^{\infty} \mu_F(A_i) : E \subseteq \cup_{i=1}^{\infty} A_i; A_i \in \mathcal{M}_{\mu_F^*}\right\}$$

Proof: HW3 Q2

□

Lemma

Given $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous, $\forall E \subseteq \mathcal{M}_{\mu_F^*}$,

$$\mu_F^*(E) = \mu_F(E) = \inf\left\{\sum_{i=1}^{\infty} \mu_F((a_i, b_i)) : E \subseteq \cup_{i=1}^{\infty} (a_i, b_i)\right\}.$$

Proof: ($\leq$) We have this by the previous proposition, since there we are taking the inf over a larger set.

($\geq$) Without loss of generality assume $\mu_F(E) < +\infty$. Fix $\varepsilon > 0$. By definition of μ_F^*, $\exists \{(a_i, b_i]\}_{i=1}^{\infty}$ with $a_i \leq b_i$ and $E \subseteq \cup_{i=1}^{\infty} (a_i, b_i]$ such that

$$\mu_F(E) + \varepsilon = \mu_F^*(E) + \varepsilon \geq \sum_{i=1}^{\infty} F(b_i) - F(a_i) = \sum_{i=1}^{\infty} \mu_F((a_i, b_i]).$$

By continuity from above, we have

$$\lim_{n\to\infty} \mu_F\left(\left(a_i, b_i + \frac{1}{n}\right)\right) = \mu_F((a_i, b_i]).$$

Notice that $\mu_F((a_i, b_i + 1)) \leq \mu_F((a_i, b_i + 1]) = F(b_i + 1) - F(a_i) < +\infty$. Further, since F is right continuous at b_i, we have $F(b_i + \delta) \to F(b_i)$ as $\delta \to 0^+$. So

$$\mu_F((a_i, b_i + \delta)) = F(b_i + \delta) - F(a_i) \to F(b_i) - F(a_i) = \mu((a_i, b_i])$$

as $\delta \to 0^+$.

So $\forall i \in \mathbb{N} \exists \delta_i > 0$ such that

$$\begin{aligned} \mu_F((a_i, b_i + \delta_i)) &= \mu_F((a_i, b_i]) + \mu_F((b_i, b_i + \delta_i]) \\ &= \mu_F((a_i, b_i]) + F(b_i + \delta_i) - F(b_i) \\ &\leq \mu_F((a_i, b_i]) + \frac{\varepsilon}{2^i}. \end{aligned}$$

Thus $E \subseteq \cup_{i=1}^{\infty} (a_i, b_i + \delta_i)$ and

$$\begin{aligned} \mu_F(E) + \varepsilon &\geq \sum_{i=1}^{\infty} \left[\mu_F((a_i, b_i + \delta_i)) - \frac{\varepsilon}{2^i}\right] \\ &= \sum_{i=1}^{\infty} \mu_F((a_i, b_i + \delta_i)) - \varepsilon. \end{aligned}$$

So,

$$\mu_F(E) + 2\varepsilon \geq \inf\left\{\sum_{i=1}^{\infty} \mu_F((a_i, b_i)) : E \subseteq \cup_{i=1}^{\infty} (a_i, b_i), a_i \leq b_i\right\}$$

Sending $\varepsilon \to 0$ gives the result.

□

Definition: Outer Regular and Inner Regular Measures

Let (X, τ) be a topological space and let Σ be a σ-algebra on X. Let μ be a measure on (X, Σ). A measurable subset $E \subseteq X$ is called **inner regular** if

$$\mu(A) = \sup\{\mu_F(K) : K \subseteq E, K \text{ compact and measurable}\}.$$

It's called inner regular because we are "approximating from within." A measurable subset $E \subseteq X$ is called **outer regular** if

$$\mu(A) = \inf\{\mu_F(U) : E \subseteq U, U \text{ open and measurable}\}$$

If a measure is both inner regular and outer regular, it is called **regular**.

Definition: Radon Measure

A measure μ on $(X, \mathcal{B}(X))$ is a **Radon measure** if it is

(i) Finite on compact sets
(ii) Outer regular
(iii) Inner regular

Theorem: Regularity of the Lebesgue-Stieltjes Measure

Suppose $F : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, right continuous function. For $E \in \mathcal{M}_{\mu_F^*}$, we have

$$\begin{aligned} (*) \quad \mu_F(E) &= \inf\{\mu_F(U) : E \subseteq U, U \text{ open}\} \\ (**) \quad\quad &= \sup\{\mu_F(K) : K \subseteq E, K \text{ compact}\}. \end{aligned}$$

That is, the Lebesgue-Stieltjes Measure is regular.

Proof: We begin with (*). Fix $E \in \mathcal{M}_{\mu_F^*}$.

$(\leq)$ If we take U open such that $E \subseteq U$, we have $\mu_F(E) \leq \mu_F(U)$ by monotonicity, so

$$\mu_F(E) \leq \inf\{\mu_F(U) : E \subseteq U, U \text{ open}\}.$$

$(\geq)$ Without loss of generality, assume $\mu_F(E) < +\infty$. Fix $\varepsilon > 0$. By the previous lemma, we have $\exists\{(a_i, b_i)\}_{i=1}^{\infty}$ with $a_i \leq b_i$ and $E \subseteq \cup_{i=1}^{\infty}$ such that

$$\mu_F(E) + \varepsilon \geq \sum_{i=1}^{\infty} \mu_F((a_i, b_i)) \geq \mu_F(\cup_{i=1}^{\infty} (a_i, b_i)) \geq \inf\{\mu_F(U) : E \subseteq U, U \text{ open}\}.$$

Now take $\varepsilon \to 0$ and we're done.

Now we do (**). By monotonicity again, $(\geq)$ is trivial, so we need only show $(\leq)$.

- *Case 1:* Assume $E \subseteq \mathbb{R}$ is bounded. Fix $\varepsilon > 0$. By (*), $\exists U \supseteq \overline{E} \setminus E$, with U open such that

$$\mu_F(U) \leq \mu_F(\overline{E} \setminus E) + \varepsilon.$$

Let $K := \overline{E} \setminus U = \overline{E} \cap U^c$, so K is closed. Notice since $\overline{E}$ is bounded, then K is bounded. Now by the Heine-Borel Theorem, K is compact. Also, $K \subseteq E$ since

$$K := \overline{E} \cap U^c \subseteq \overline{E} \cap \left(\overline{E} \setminus E\right)^c = \overline{E} \cap \left(\overline{E} \cap E^c\right)^c = \overline{E} \cap \left(\overline{E}^c \cup E\right) = E.$$

Further,

$$\mu_F(E) = \mu_F(E \cap U) + \mu_F(E \setminus U) \leq \mu_F(E \cap U) + \mu_F(K).$$

Thus

$$\begin{aligned} \mu_F(K) &\geq \mu_F(E) - \mu_F(E \cap U) \\ &= \mu_F(E) - [\mu_F(U) - \mu_F(U \setminus E)] \\ &\geq \mu_F(E) - \mu_F(U) + \mu_F(\overline{E} \setminus E) \\ &\geq \mu_F(E) - \varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0$ we obtain $(\leq)$, showing E is inner regular.

- *Case 2:* Assume $E \subseteq \mathbb{R}$ is unbounded. Define $E_j = E \cap (j, j+1]$ with $j \in \mathbb{Z}$. By what we've already shown, $\exists K_j \subseteq E_j$ with K_j compact such that

$$\mu_F(K_j) \geq \mu_F(E_j) - \frac{\varepsilon}{2^{|j|}}.$$

Now let $H_n := \cup_{j=-n}^n K_j$, which we note is a disjoint union. Observe H_n is compact and $H_n \subseteq E$. For all $n \in \mathbb{N}$,

$$\mu_F(H_n) = \sum_{j=-n}^{n} \mu_F(K_j) \geq \sum_{j=-n}^{n} \left[\mu_F(E_j) - \frac{\varepsilon}{2^{|j|}}\right] \geq \mu_F\left(\cup_{j=-n}^n E_j\right) - 3\varepsilon.$$

By continuity from below,

$$\lim_{n \to +\infty} \mu_F\left(\bigcup_{j=-n}^{n} E_j\right) = \mu_F(E).$$

Now if $\mu_F(E) < +\infty$, $\exists N \in \mathbb{N}$ such that $n \geq N$ gives

$$\begin{aligned} \sup\{\mu_F(K) : K \subseteq E, K \text{ compact}\} &\geq \mu(H_n) \\ &\geq \mu_F\left(\cup_{j=-n}^n E_j\right) - 3\varepsilon \\ &\geq \mu_F(E) - 4\varepsilon. \end{aligned}$$

On the other hand if $\mu_F(E) = +\infty$,

$$\lim_{n \to +\infty} \mu_F\left(\bigcup_{j=-n}^{n} E_j\right) = +\infty \implies \sup\{\mu_F(K) : K \subseteq E, K \text{ compact}\} = +\infty \geq \mu_F(E).$$

So either way we get the result.

□

4 Integration

4.A Measurable Functions

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Proposition

Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. Then the following are σ-algebras:

- $\{f^{-1}(E) : E \in \mathcal{N}\}$, which we call the **pullback** of $\mathcal{N}$
- $\{E : f^{-1}(E) \in \mathcal{M}\}$, which we call the **push forward** of $\mathcal{M}$

Definition: $(\mathcal{M}, \mathcal{N})$-measurable

$f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if, for all $E \in \mathcal{N}$, we have $f^{-1}(E) \in \mathcal{M}$, i.e., the preimage of every measurable set is measurable. Note that, we can also write this in terms of the pullback and push forward:

$$f : X \to Y \text{ is } (\mathcal{M}, \mathcal{N})\text{-measurable} \iff \{f^{-1}(E) : E \in \mathcal{N}\} \subseteq \mathcal{M}$$
$$\iff \{E : f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{N}$$

Importantly, when $f : X \to \mathbb{R}$ (resp $\overline{\mathbb{R}}$), we can assume the codomain is endowed with $\mathcal{B}_{\mathbb{R}}$ (resp $\mathcal{B}_{\overline{\mathbb{R}}}$).

Definition: Lebesgue and Borel Measurable Function

We call $f : \mathbb{R} \to \mathbb{R}$ **Lebesgue measurable** if it is $(\mathcal{M}_{\lambda^*}, \mathcal{B}_{\mathbb{R}})$-measurable.

Given X, Y topological spaces, $f : X \to Y$ is called **Borel measurable** if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Remark

Given $f : \mathbb{R} \to \mathbb{R}$, note that f is Borel measurable $\implies f$ is Lebesgue measurable.

However, $\mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{M}_{\lambda^*}$, so the reverse is not true. Thus to be more general, we'll consider f to be Lebesgue measurable in the general case in many of the following results.

Proposition

Given measurable spaces $(X, \mathcal{M}), (Y, \mathcal{N})$ where $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $\forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{M}$.

Proof: $(\implies)$ is immediate since $\mathcal{E} \subseteq \mathcal{N}$.

$(\impliedby)$: Note that $\mathcal{E} \subseteq \{E : f^{-1}(E) \in \mathcal{M}\}$. Since $\mathcal{N}$ is the smallest σ-algebra containing $\mathcal{E}$, we have $\mathcal{N} \subseteq \{E : f^{-1}(E) \in \mathcal{M}\}$.

□

Corollary

If X, Y topological spaces, then every continuous function $f : X \to Y$ is Borel measurable.

Proof: Since f is continuous, $\forall U \subseteq Y$, $f^{-1}(U)$ is open, so $f^{-1}(U) \in \mathcal{B}_X$.

□

Corollary

If $(X, \mathcal{M})$ is a measure space and $f : X \to \overline{\mathbb{R}}$ then

$$f \text{ is } (\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})\text{-measurable} \iff f^{-1}((a, +\infty]) \in \mathcal{M} \forall a \in \mathbb{R}.$$

Proof: We showed $\{(a, +\infty] : a \in \mathbb{R}\}$ generate $\mathcal{B}_{\mathbb{R}}$. Then this follows from the proposition.

□

Proposition

Suppose $(X, \mathcal{M})$ is a measure space and let $f : X \to \overline{\mathbb{R}}$ be given by $f(x) = c \forall x \in X$ for some $c \in \overline{\mathbb{R}}$. Then f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Proof: Given $E \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have

$$f^{-1}(E) = \begin{cases} \emptyset & \text{if } c \notin E \\ X & \text{if } c \in E \end{cases}$$

Now the result follows from the fact that any σ-algebra contains $\emptyset$ and X.

□

Theorem

Fix a measurable space $(X, \mathcal{M})$ and a sequence $\left\{f_i : X \to \overline{\mathbb{R}}\right\}_{i=1}^{\infty}$ with each f_i being $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. Then the following are also $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable:

(i) $f_1 + f_2$
(ii) $f_1 f_2$
(iii) $f_1 \vee f_2$, where $(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\}$
(iv) $f_1 \wedge f_2$, where $(f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\}$
(v) $\sup_i f_i$
(vi) $\inf_i f_i$
(vii) $\limsup_{i \to +\infty} f_i$
(viii) $\liminf_{i \to +\infty} f_i$
(ix) $\lim_{i \to +\infty} f_i$ (assuming the limit exists)

Proof:

(i) HW 4

(ii) HW4

(iii) Note that

$$\begin{aligned}(f_1 \vee f_2)^{-1}((a,+\infty]) &= \{x \in X : (f_1 \vee f_2)(x)\} \\ &= \{x \in X : f_1(x) > a\} \cup \{x : X : f_2(x) > a\} \\ &= f_1^{-1}((a,+\infty]) \cup f_2^{-1}((a,+\infty]) \in \mathcal{M}\end{aligned}$$

(iv) We have $f_1 \wedge f_2 = -((-f_1) \vee (-f_2))$, so this reduces to (iii)

(v) We have

$$\begin{aligned}\left(\sup_i f_i\right)^{-1}((a,+\infty]) &= \left\{x \in X : \sup_i f_i(x) > a\right\} \\ &= \{x \in X : \exists i \text{ such that } f_i(x) > a\} \\ &= \cup_{i=1}^{\infty} \{x \in X : f_i(x) > a\} \\ &= \cup_{i=1}^{\infty} f_i^{-1}((a,+\infty]) \in \mathcal{M}\end{aligned}$$

(vi) Follows from (v)

(vii) Note $\limsup_{i\to+\infty} = \inf_{n\in\mathbb{N}} \sup_{i\geq N} f_i$

(viii) Follows from (vii)

(ix) $\lim_{i\to+\infty} = \limsup f_i$ whenever the limit exists

□

Remark

Consider $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Then if both are Borel measurable, notice $f \circ g$ is Borel measurable. Also if f is Lebesgue measurable and g is Borel measurable, then $f \circ g$ is Lebesgue measurable, but $g \circ f$ isn't.

4.B Simple Functions

Definition

For any $A \subseteq X$, define the **indicator function** of that set A by

$$1_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ otherwise} \end{cases}$$

Note that for any B in the codomain,

$$1_A^{-1}(B) = \begin{cases} A & \text{if } 1 \in B, 0 \notin B \\ A^c & \text{if } 1 \notin B, 0 \in B \\ \emptyset & \text{if } 0, 1 \notin B \\ X & \text{if } 0, 1 \in B \end{cases}$$

Note if $A \in \mathcal{M}$, 1_A is a $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable function.

Definition: Simple Function and Standard Representation

A $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable function $f : X \to \mathbb{R}$ is a **simple function** if $f(X)$ is a finite set.

In particular, this means we can represent a simple function f by

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$$

where $f(X) = \{c_1, ..., c_n\}$ and $E_i = f^{-1}(c_i)$. This is called the **standard representation** of that function.

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Remark

In the above, notice $\{E_i\}_{i=1}^{\infty}$ are disjoint and form a partition of the domain.

Example

Consider $f(x) = 2$ and $X = \mathbb{R}$. Then $f(x) = 2 \cdot \mathbb{1}_{\mathbb{R}}(x)$.

Definition: Integral of a Simple Function

For a measure space $(X, \mathcal{M}, \mu)$ and nonnegative simple function f with standard representation

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$$

we define $\int f \mathrm{d}\mu := \sum_{i=1}^{n} c_i \mu(E_i)$.

Remark

Suppose for a moment that we allowed negative simple functions. Consider

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, +\infty) \\ -1 & \text{if } x \in (-\infty, 0) \end{cases}$$

Then $\int f \, \mathrm{d}\lambda = 1 \cdot \lambda([0, +\infty)) + (-1)\lambda((-\infty, 0)) = \infty - \infty$.

Note the problem this presents! It's unclear how to define $\infty - \infty$. Thus for now, we will ignore negative functions.

Notation

We will use the following notation for integrals:

$$\int_X f \, \mathrm{d}\mu = \int_X f(x) \mathrm{d}\mu(x) = \int f(x) \mathbb{1}_X \mathrm{d}\mu(x).$$

We call this integrating with respect to the measure μ on a set X.

Proposition

Given a measure space $(X, \mathcal{M}, \mu)$ and simple functions $f, g : X \to [0, +\infty)$,

(i) If $c \geq 0$, $\int cf \mathrm{d}\mu = c \int f \mathrm{d}\mu$
(ii) $\int (f+g) \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu$
(iii) $f \leq g \Rightarrow \int f \mathrm{d}\mu \leq \int g \mathrm{d}\mu$
(iv) The function $A \mapsto \int_A f \mathrm{d}\mu$ is a measure on $(X, \mathcal{M})$

Proof:

(i) Without loss of generality, suppose $c > 0$. Then $cf = \sum_{i=1}^{n} ca_i \mathbb{1}_{E_i}$, so we have

$$\int cf \mathrm{d}\mu = \sum_{i=1}^{n} ca_i \mu(E_i) = c \sum_{i=1}^{n} a_i \mu(E_i) = c \int f \mathrm{d}\mu.$$

(ii) Suppose $\{E_i\}_{i=1}^{n}$ and $\{F_j\}_{j=1}^{m}$ are disjoint partitions of X with

$$\int f \mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(E_i) \qquad \text{and} \qquad \int g \mathrm{d}\mu = \sum_{j=1}^{m} b_j \mu(F_j).$$

Observe we can write

$$E_i = \bigcup_{j=1}^{m} E_i \cap F_j \qquad F_j = \bigcup_{i=1}^{n} F_j \cap E_i$$

where these are both disjoint unions (which will allow us to get measure equality). Also, observe $(f+g)(X) = \{c_1, ..., c_\ell\}$, and we must have $c_k = a_i + b_j$ for some i, j. Therefore, we can say

$$\begin{aligned}
\int f \mathrm{d}\mu + \int g \mathrm{d}\mu &= \sum_{i=1}^{n} a_i \mu(E_i) + \sum_{j=1}^{m} b_j \mu(F_j) \\
&= \sum_{i,j} a_i \mu(E_i \cap F_j) + \sum_{i,j} b_j \mu(E_i \cap F_j) \\
&= \sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j) \\
&= \sum_{k=1}^{\ell} \sum_{i,j : a_i + b_j = c_k} (a_i + b_j) \mu(E_i \cap F_j) \\
&= \sum_{k=1}^{\ell} c_k \mu\left(\bigcup_{i,j : a_i + b_j = c_k} E_i \cap F_j \right) \\
&= \sum_{k=1}^{\ell} c_k \mu(G_k) \quad \text{where } G_k := (f+g)^{-1}(c_k) \\
&= \int (f+g) \mathrm{d}\mu.
\end{aligned}$$

(iii) Suppose $f \leq g$. Then whenever $E_i \cap F_j \neq \emptyset$, we have $a_i \leq b_j$. Therefore,

$$\int f \mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(E_i) = \sum_{i,j} a_i \mu(E_i \cap F_j) \leq \sum_{i,j} b_j \mu(E_i \cap F_j) = \sum_{j=1}^{m} b_j \mu(F_j) = \int g \mathrm{d}\mu.$$

(iv) Define $\nu(A) := \int_A f \mathrm{d}\mu$. Notice this is a nonnegative function on $\mathcal{M}$. Now we show ν is a measure.

First, observe $\nu(\emptyset) = \int_\emptyset f \mathrm{d}\mu = \int f \cdot \mathbb{1}_\emptyset \mathrm{d}\mu = 0$.

Now let $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}$ be disjoint, and let $A := \cup_{k=1}^\infty A_k$. Then we have that

$$\begin{aligned}
\nu(A) &= \int_A f \mathrm{d}\mu \\
&= \int f \mathbb{1}_A \mathrm{d}\mu \\
&= \int \left(\sum_{i=1}^n a_i \mathbb{1}_{E_i} \right) \mathbb{1}_A \mathrm{d}\mu \\
&= \sum_{i=1}^n a_i \int \mathbb{1}_{E_i} \mathbb{1}_A \mathrm{d}\mu \\
&= \sum_{i=1}^n a_i \mu(E_i \cap A) \\
&= \sum_{i=1}^n a_i \sum_{k=1}^\infty \mu(E_i \cap A_k) \\
&= \sum_{k=1}^\infty \sum_{i=1}^n a_i \mu(E_i \cap A_k) \\
&= \sum_{k=1}^\infty \int \sum_{i=1}^n a_i \mathbb{1}_{E_i \cap A_k} \mathrm{d}\mu \\
&= \sum_{k=1}^\infty \int_{A_k} f \mathrm{d}\mu \\
&= \sum_{i=1}^\infty \nu(A_k).
\end{aligned}$$

□

Remark

Note that (i) and (ii) ensure that the definition of $\int f \mathrm{d}\mu$ is independent of the representation of f as a nonnegative linear combination of simple functions.

That is, if

$$f = \underbrace{\sum_{i=1}^{n} c_i \mathbb{1}_{E_i}}_{\text{representation 1}} = \underbrace{\sum_{j=1}^{m} d_j \mathbb{1}_{F_j}}_{\text{representation 2}},$$

then we have

$$\begin{aligned}
&\int \left(\sum_{i=1}^{n} c_i \mathbb{1}_{E_i} - \sum_{j=1}^{m} d_j \mathbb{1}_{F_j} \right) \mathrm{d}\mu = \int (f - f) \mathrm{d}\mu = 0 \\
&\implies \sum_{i=1}^{n} c_i \int \mathbb{1}_{E_i} \mathrm{d}\mu = \sum_{j=1}^{m} d_j \int \mathbb{1}_{F_j} \mathrm{d}\mu \\
&\implies \sum_{i=1}^{n} c_i \mu(E_i) = \sum_{j=1}^{m} d_j \mu(F_j).
\end{aligned}$$

4.C Integration of Nonnegative Measurable Functions

Definition: Integral of Nonnegative Measurable Function

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f : X \to [0, +\infty]$ is a measurable function. Then we define

$$\boxed{\int f \mathrm{d}\mu := \sup \left\{ \int \varphi \mathrm{d}\mu : 0 \le \varphi \le f; \varphi \text{ simple} \right\}}$$

as the integral of f with respect to μ.

Remark

(i) Note that if f is simple, this is the same as the previous definition.

(ii) If $c = 0$, then $cf\mathrm{d}\mu = c\int f\mathrm{d}\mu$. If $c > 0$, then

$$\begin{aligned}\int cf\mathrm{d}\mu &= \sup\left\{\int \varphi\mathrm{d}\mu : 0 \le \varphi \le cf; \varphi \text{ simple}\right\}\\ &= \sup\left\{\int \varphi\mathrm{d}\mu : 0 \le \frac{\varphi}{c} \le f; \varphi \text{ simple}\right\}\\ &= \sup\left\{c\int \psi\mathrm{d}\mu : 0 \le \psi \le f; \psi \text{ simple}\right\}\\ &= c\sup\left\{\int \psi\mathrm{d}\mu : 0 \le \psi \le f; \psi \text{ simple}\right\}\\ &= c\int f\mathrm{d}\mu.\end{aligned}$$

(iii) If $f \le g$, then $\int f\mathrm{d}\mu \le \int g\mathrm{d}\mu$.

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Theorem: Monotone Convergence Theorem

Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable such that $f_n \le f_{n+1} \forall n$, then

$$\lim_{n\to\infty}\int f_n\mathrm{d}\mu = \int \lim_{n\to\infty} f_n\mathrm{d}\mu.$$

Proof: Note that both limits exist by monotonicity.

($\le$) By hypothesis, $f_n \le \lim_{n\to\infty} f_n$, so $\int f_n\mathrm{d}\mu \le \int \lim_{n\to\infty} f_n\mathrm{d}\mu$.

($\ge$) Let $\varphi : X \to [0, +\infty)$ be a simple function such that $0 \le \varphi \le \lim_{n\to\infty} f_n$. Without loss of generality take $\lim_{n\to+\infty}\int f_n\mathrm{d}\mu < +\infty$. Take $a \in (0,1)$. Note that if $\varphi(x) \ne 0$, then $a\varphi(x) < \lim_{n\to\infty} f_n(x)$. Now define $E_n := \{x : f_n(x) \ge a\varphi(x)\}$. Note this is measurable since it is the preimage of the Borel measurable set $[0, +\infty)$.

Since $f_n \le f_{n+1}$, we have $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$ Further, $\cup_{n=1}^{\infty} E_n = X$ because

- if $\varphi(x) = 0$, then $x \in E_n \forall n \in \mathbb{N}$
- if $\varphi(x) > 0$ then $\exists N$ such that $n \ge N \Rightarrow x \in E_n$

Therefore,

$$\int f_n\mathrm{d}\mu \ge \int_{E_n} f_n\mathrm{d}\mu \ge \int_{E_n} a\varphi\mathrm{d}\mu = a\int_{E_n}\varphi\mathrm{d}\mu.$$

Now since we showed in the previous proposition that $A \mapsto \int_A \varphi\mathrm{d}\mu$ is a measure, by continuity from below we have

$$\lim_{n\to\infty}\int f_n\mathrm{d}\mu \ge a\lim_{n\to\infty}\int_{E_n}\varphi\mathrm{d}\mu = a\int\varphi\mathrm{d}\mu.$$

Now take $a \to 1$.

□

Theorem

Given $f : X \to [0, +\infty]$ measurable, there exists a seqeunce f_n of nonnegative simple functions such that $f_n \nearrow f$ pointwise.

Proof: For $n \in \{0\} \cup \mathbb{N}$, we chop up the range of f, up to height 2^n, in increments of height 2^{-n}. In particular, define

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}]); \qquad F_n = f^{-1}((2^n, +\infty]); \qquad k = 0, ..., 2^{2n-1}$$

and

$$f_n = \sum_{k=0}^{2^{2n-1}} k2^{-n} \mathbb{1}_{E_n^k} + 2^n \mathbb{1}_{F_n}.$$

Some important properties are that:

- $f_n(x) \leq f_{n+1}(x) \forall x \in X$
- $0 \leq f(x) - f_n(x) \forall x \in X$
- $0 \leq f(x) - f_n(x) \leq 2^{-n} \forall x \in F_n^c$

Notice that if $x \in \cup_{i=1}^{\infty} F_n^c$ then $\exists N \in \mathbb{N}$ such that $x \in F_n^c \forall n \geq N$ we have

$$0 \leq f(x) - f_n(x) \leq 2^{-n} \forall n \geq N$$

so $\lim_{n\to\infty} f_n(x) = f(x)$. On the other hand, if $x \in \cap_{n=1}^{\infty} F_n$ then $f(x) = +\infty$ and $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2^n = +\infty = f(x)$

□

Theorem: Beppo-Levi

Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable functions, then

$$\sum_{n=1}^{\infty} \int f_n \mathrm{d}\mu = \int \sum_{n=1}^{\infty} f_n \mathrm{d}\mu.$$

Proof: First, fix $f, g : X \to [0, +\infty]$ measurable. By the previous theorem, $\exists \{\varphi_i\}_{i=1}^{\infty}, \{\psi_i\}_{i=1}^{\infty}$ such that $\varphi_i \nearrow f, \psi_i \nearrow g$ pointwise. So $\varphi_i + \psi_i \nearrow f + g$ pointwise. Therefore

$$\begin{aligned} \int (f+g)\mathrm{d}\mu &= \lim_{i\to\infty} \varphi_i + \psi_i \mathrm{d}\mu \\ &= \lim_{i\to\infty} \int \varphi_i + \psi_i \mathrm{d}\mu \qquad \text{(by MCT)} \\ &= \lim_{i\to\infty} \int \varphi_i \mathrm{d}\mu + \int \psi_i \mathrm{d}\mu \\ &= \lim_{i\to\infty} \int \varphi_i \mathrm{d}\mu + \lim_{i\to\infty} \int \psi_i \mathrm{d}\mu \\ &= \int f \mathrm{d}\mu + \int g \mathrm{d}\mu. \end{aligned}$$

By induction, $\forall N \in \mathbb{N}$, we have

$$\int \sum_{i=1}^{N} f_n \mathrm{d}\mu = \sum_{n=1}^{N} \int f_n \mathrm{d}\mu.$$

By the Monotone Convergence Theorem,

$$\sum_{n=1}^{\infty} \int f_n \mathrm{d}\mu = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \mathrm{d}\mu \overset{\text{MCT}}{=} \int \sum_{n=1}^{\infty} f_n \mathrm{d}\mu.$$

□

Now, we introduce one more theorem on interchanging limits and integrals without the monotonicity requirement.

Theorem: Fatou's Lemma

Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable,

$$\liminf_{n\to\infty} \int f_n \mathrm{d}\mu \geq \int \liminf_{n\to\infty} f_n \mathrm{d}\mu.$$

Proof: By definition, $\liminf_{n\to\infty} f_n = \lim_{k\to\infty} \inf_{n\geq k} f_n = \lim_{k\to\infty} g_k$. Then by the MCT,

$$\lim_{k\to\infty} \int g_k \mathrm{d}\mu \overset{\text{MCT}}{=} \int \lim_{k\to\infty} g_k \mathrm{d}\mu = \int \liminf_{n\to\infty} f_n \mathrm{d}\mu.$$

By definition, $g_k \leq f_k \forall k \in \mathbb{N}$, so

$$\liminf_{n\to\infty} \int f_n \mathrm{d}\mu \geq \liminf_{n\to\infty} \int g_n \mathrm{d}\mu = \int \liminf_{n\to\infty} f_n \mathrm{d}\mu.$$

□

Example: Strict Inequality in Fatou's Lemma

Take our measure space to be $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$.

(i) *Run away to infinity*: Suppose $f_n = \mathbb{1}_{[n,n+1]}$. Observe $\lim_{n\to\infty} f_n = 0$ pointwise. But

$$1 = \lim_{n\to\infty} \int f_n \, \mathrm{d}\lambda > \int \lim_{n\to\infty} f_N \, \mathrm{d}\lambda = 0.$$

(ii) *Goes up the spout*: Let

$$f_n = n\mathbb{1}_{[0,\frac{1}{n}]}; \qquad \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if} \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}.$$

Then

$$1 = \lim_{n\to\infty} \int f_n \, \mathrm{d}\lambda > \int \lim_{n\to\infty} f_n \, \mathrm{d}\lambda = 0.$$

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Notice that we haven't justified that last inequality yet. We do so in the following proposition:

Proposition

Given $f : X \to [0, +\infty]$ measurable, we have

$$\int f \mathrm{d}\mu = 0 \iff f = 0 \quad \mu\text{-almost everywhere, i.e. } \mu(\{x : f(x) \neq 0\}) = 0.$$

Proof: First suppose f is simple, i.e.,

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}, \qquad a_i \geq 0.$$

Then

$$\begin{aligned}\int f \mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(E_i) = 0 &\iff \forall i, \text{either } a_i = 0 \text{ or } \mu(E_i) = 0 \\ &\iff f = 0 \ \ \mu\text{-a.e.}\end{aligned}$$

Now consider a general $f : X \to [0, +\infty]$.

$(\Longleftarrow)$ By definition,

$$\int f \mathrm{d}\mu = \sup\left\{ \int \varphi \mathrm{d}\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = \sup\{0\} = 0.$$

$(\Longrightarrow)$ We proceed by contraposition. Assume $f = 0$ is not μ-a.e. Note that $\{x : f(x) > 0\} = \cup_{n=1}^{\infty} \left\{x : f(x) > \frac{1}{n}\right\}$. By countable subadditivity,

$$0 < \mu(\{x : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \mu\left(\left\{x : f(x) > \frac{1}{n}\right\}\right).$$

So $\exists n \in \mathbb{N}$ such that $\mu\left(\left\{x : f(x) > \frac{1}{n}\right\}\right) > 0$. Let $\varphi = \frac{1}{n} \mathbb{1}_{\{x : f(x) > \frac{1}{n}\}}$. Then $\varphi \leq f$. By definition,

$$\int f \mathrm{d}\mu \geq \int \varphi \mathrm{d}\mu = \frac{1}{n} \mu\left(\left\{x : f(x) > \frac{1}{n}\right\}\right) > 0.$$

□

4.D Integration of Real-Valued Measurable Functions

Suppose we are in a measure space $(X, \mathcal{M}, \mu)$. Given $f : X \to \overline{\mathbb{R}}$, define the "positive part" to be $f_+ := f \vee 0$ and the "negative part" to be $f_- := (-f) \vee 0$. Therefore, $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Definition: Integrable Real Valued Function

Given $f : X \to \overline{\mathbb{R}}$ measurable, if either $\int f_+ \mathrm{d}\mu < +\infty$ or $\int f_- \mathrm{d}\mu < +\infty$, then define

$$\int f \mathrm{d}\mu := \int f_+ \mathrm{d}\mu - \int f_- \mathrm{d}\mu.$$

If both $\int f_+ \mathrm{d}\mu < +\infty$ and $\int f_- \mathrm{d}\mu < +\infty$, then we say f is **integrable** and write $f \in L^1(\mu)$. Note that this means the integral can be defined even if f is not integrable.

Remark

Observe that

$$\begin{aligned} f \text{ integrable} &\iff \int f_+ \mathrm{d}\mu < +\infty \text{ and } \int f_- \mathrm{d}\mu < +\infty \\ &\iff \int f_+ \mathrm{d}\mu + \int f_- \mathrm{d}\mu < +\infty \\ &\iff \int |f| \mathrm{d}\mu < +\infty. \end{aligned}$$

Proposition

$L^1(\mu)$ is a real vector space and $f \mapsto \int f \mathrm{d}\mu$ is a linear functional on $L^1(\mu)$.

Proof: First we show this is a vector space. Let $f, g \in L^1(\mu)$ and $a, b \in \mathbb{R}$. Then observe

$$\int |af + bg| \mathrm{d}\mu \leq \int |a| \cdot |f| + |b| \cdot |g| \mathrm{d}\mu = |a| \int |f| \mathrm{d}\mu + |b| \int |g| \mathrm{d}\mu < +\infty.$$

Now we check that integration is a linear functional. Fix $f \in L^1(\mu)$ with $a \geq 0$. Then $\int af \mathrm{d}\mu = a \int f \mathrm{d}\mu$. Now in the $a < 0$ case, note $af = (-a)(-f)$, so the result follows. Finally, for any $f, g \in L^1(\mu)$, we have

$$\begin{aligned} \int (f+g) \mathrm{d}\mu &= \int (f+g)_+ \mathrm{d}\mu - \int (f+g)_- \mathrm{d}\mu \\ &= \int f_+ \mathrm{d}\mu + \int g_+ \mathrm{d}\mu - \int f_- \mathrm{d}\mu - \int g_- \mathrm{d}\mu \\ &= \int f \mathrm{d}\mu + \int g \mathrm{d}\mu. \end{aligned}$$

We justify the second equality by noting

$$\begin{aligned} & (f+g)_+ - (f+g)_- = (f_+ - f_-) + (g_+ - g_-) \\ \implies & (f+g)_+ + f_- + g_- = f_+ + g_+ + (f+g)_- \end{aligned}$$

so by Beppo-Levi,

$$\int (f+g)_+ \mathrm{d}\mu + \int f_- \mathrm{d}\mu + \int g_- \mathrm{d}\mu = \int (f+g)_- \mathrm{d}\mu + \int f_+ \mathrm{d}\mu + \int g_+ \mathrm{d}\mu.$$

□

Remark

Note that some steps in the preceding proof were not properly justified, since we haven't yet shown $f \in L^1(\mu)$ implies

$$\mu(\{x : f(x) \in \{-\infty, +\infty\}\}) = 0,$$

so that the values of the function don't affect the integral value. Thus, we could have some $\infty - \infty$ expressions above. We will show this shortly.

Proposition

If $f \in L^1(\mu)$, then $\left|\int f \mathrm{d}\mu\right| \leq \int |f| \mathrm{d}\mu$.

Proof: Simply observe

$$\left|\int f \mathrm{d}\mu\right| \leq \int f_+ \mathrm{d}\mu + \int f_- \mathrm{d}\mu = \int |f| \mathrm{d}\mu.$$

□

Now, it would be nice if we could show $L^1(\mu)$ is a normed vector space, with norm

$$\|f - g\|_{L^1(\mu)} := \int |f - g| \mathrm{d}\mu.$$

The problem is that based on the theory laid out thus far, the norm would be degenerate, meaning $\exists f, g$ with $f \neq g$ such that $\|f - g\|_{L^1(\mu)} = 0$.

Corollary

If $f, g \in L^1(\mu)$ then $\int |f - g| \mathrm{d}\mu = 0 \Leftrightarrow f = g$ μ-a.e.

Proof: ($\Longleftarrow$) If $f = g$ μ-a.e., then $|f - g| = 0$ μ-a.e., so

$$\int |f - g| \mathrm{d}\mu = 0.$$

($\Longrightarrow$) Suppose $\int |f - g| \mathrm{d}\mu = 0$. Since $|f - g| \geq 0$, we must have $|f - g| = 0$ μ-a.e, and thus $f = g$ μ-a.e.

□

Importantly, this corollary tells us that if an integrable function is modified on a null set, then it doesn't change the integral. Further, even if a function f is only defined μ-.a.e, $\int f \mathrm{d}\mu$ is still well-defined. This motivates an improved definition for $L^1(\mu)$.

Definition: $L^1(\mu)$

We define

$$L^1(\mu) := \left\{ f : X \to \overline{\mathbb{R}} \text{ measurable}, \int |f| < +\infty \right\} / \sim$$

where $f \sim g$ if $f = g$ μ-a.e.

Remark

By abuse of notation, we will let $f \in L^1(\mu)$ denote

- the equivalence class
- a representative of the equivalence class
- a representative that is only defined μ-a.e.

Proposition

$\|f\|_{L^1(\mu)} := \int |f| \mathrm{d}\mu$ is a norm on $L^1(\mu)$.

Proof: nondegeneracy, triangle inequality, positive homogeneity

□

4.E Dominated Convergence Theorem

Now, we move onto results that allow us to interchange limits and integrals for real valued functions.

Theorem: Dominated Convergence Theorem

Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ such that $\lim_{n\to\infty} f_n$ exists μ-a.e., if $\exists g \in L^1(\mu)$ such that $|f_n| \leq g \forall n \in \mathbb{N}$ μ-a.e., then

$$\lim_{n\to\infty} \int f_n \mathrm{d}\mu = \int \lim_{n\to\infty} f_n \mathrm{d}\mu$$

Proof: Since $\lim_{n\to\infty} f_n$ exists μ-a.e. and $|f_n| \leq g$ μ-a.e. $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} f_n \in L^1(\mu)$. (Note there is a subtlety in that the f_n's may be μ-a.e. in different places, but this is irrelevant because we can just take the countable union of those sets.)

Since $g - f_n \geq 0$, $g + f_n \geq 0$ μ-a.e., by Fatou's Lemma (can be invoked due to nonnegativity) we have

$$\begin{aligned} \int g \mathrm{d}\mu + \liminf_{n\to\infty} \int f_n = \liminf_{n\to\infty} \int g + f_n \mathrm{d}\mu &\geq \int \liminf_{n\to\infty} g + f_n \mathrm{d}\mu \\ &= \int g \mathrm{d}\mu + \int \liminf_{n\to\infty} f_n \mathrm{d}\mu. \end{aligned}$$

Likewise,

$$\begin{aligned} \int g \mathrm{d}\mu - \limsup_{n\to\infty} \int f_n \mathrm{d}\mu &= \limsup_{n\to\infty} \int g \mathrm{d}\mu - \int f_n \mathrm{d}\mu \\ &\geq \liminf_{n\to\infty} \int g - f_n \mathrm{d}\mu \\ &\geq \int \liminf_{n\to\infty} g - f_n \mathrm{d}\mu \\ &= \int g \mathrm{d}\mu + \int \left[\liminf_{n\to\infty} (-f_n)\right] \mathrm{d}\mu \\ &= \int g \mathrm{d}\mu - \int \left[\limsup_{n\to\infty} f_n\right] \mathrm{d}\mu. \end{aligned}$$

Since $\int g < +\infty$, this gives

$$\int \lim_{n\to\infty} f_n \mathrm{d}\mu = \int \limsup_{n\to\infty} f_n \mathrm{d}\mu \geq \limsup_{n\to\infty} \int f_n \mathrm{d}\mu \geq \liminf_{n\to\infty} \int f_n \mathrm{d}\mu \geq \int \liminf_{n\to\infty} f_n \mathrm{d}\mu = \int \lim_{n\to\infty} f_n \mathrm{d}\mu.$$

Thus $\lim_{n\to\infty} \int f_n$ exists and in fact,

$$\lim_{n\to\infty}\int f_n \mathrm{d}\mu = \int \lim_{n\to\infty} f_n \mathrm{d}\mu.$$

as desired.

□

Remark

Note that the Dominated Convergence Theorem does not require that g is bounded.

In fact, g bounded $\not\Rightarrow g \in L^1(\mu)$, since we can take $g(x) = 1$ for example, and $g \in L^1(\mu) \not\Rightarrow g$ bounded, since we can for example take

$$g(x) = \begin{cases} \frac{1}{\sqrt{x}} \text{ if } x \in (0,1] \\ 0 \text{ otherwise} \end{cases}$$

Then

$$\int_{\mathbb{R}} g(x)\,\mathrm{d}\lambda = \int_{(0,1]} \frac{1}{\sqrt{x}}\,\mathrm{d}\lambda(x) = \int_0^1 \frac{1}{\sqrt{x}} = 2 < +\infty$$

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Definition: Support

Define the **support** of a function f to be the closure of the set on which f is nonzero:

$$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}.$$

Now we apply the DCT to two useful subsets of functions that are dense in $L^1(\mu)$.

Theorem

For any measure space $(X, \mathcal{M}, \mu)$, simple functions are dense in $L^1(\mu)$.

For Lebesgue-Stiltjes measures on $\mathbb{R}$, the following are also dense in $L^1(\mu)$:

(i) Simple functions of the form $\xi = \sum_{j=1}^{N} a_j \mathbb{1}_{F_j}$, $F_j = \cup_{i=1}^{m_j} I_{ij}$ for disjoint open intervals $\{I_{i,j}\}_{i=1}^{m_j}$.

(ii) $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}$ where f is continuous and $\operatorname{supp} f$ is compact$\}$

Note that by definition, supp f is closed, so by the Heine-Borel theorem, we mean by the above that the support is bounded.

Proof: Fix $f \in L^1(\mu)$. Since f_+ and f_- are nonnegative measurable functions, $\exists$ simple functions ψ_n, φ_n such that $\psi_n \nearrow f_+$ and $\varphi_n \nearrow f_-$. Further,

$$|(\psi_n - \varphi_n) - f| \overset{n\to+\infty}{\to} 0 \text{ and } |(\psi_n - \varphi_n) - f| \leq \psi_n + \varphi_n + |f| \leq 2|f|.$$

Thus by the Dominated Convergence Theorem,

$$\lim_{n\to+\infty}\int ||\psi_n - \varphi_n| - f| \mathrm{d}\mu = \int \lim_{n\to\infty} |(\psi_n - \varphi_n) - f| \mathrm{d}\mu = \int 0 \mathrm{d}\mu = 0.$$

Now suppose μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$. We will show that simple functions can be approximated by a function of the form (i) and functions of the form (i) can be approximated by $C_c(\mathbb{R})$.

Fix a simple function $\varphi \in L^1(\mu)$. We may express φ as

$$\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{E_j} \text{ where } a_j \neq 0 \forall j, \{E_j\}_{j=1}^{n} \text{ are disjoint}$$

Thus $\forall j$, we have $|a_j|\mu(E_j) \leq \int |\varphi| \mathrm{d}\mu < +\infty$ so $\mu(E_j) < +\infty \forall j$.

Now by HW4 Q2, recall that for any $E \in \mathcal{M}_{\mu_F^*}$ with $\mu_F^*(E) < +\infty$, for all $\varepsilon > 0 \exists$ disjoint open intervals $\{I^i\}_{i=1}$ such that

$$\mu_F(E \Delta \cup_{i=1}^{m} I^i) < \varepsilon.$$

Thus for all $\varepsilon > 0 \exists$ disjoint open intervals $\{I_j^i\}_{i=1}^{m_j}$ such that

$$\mu\left(E_j \Delta \cup_{i=1}^{m_j} I_j^i\right) < \frac{\varepsilon}{n \max_j |a_j|}.$$

Therefore,

$$\left\| \varphi - \sum_{j=1}^{n} a_j \mathbb{1}_{\cup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)} \leq \sum_{j=1}^{n} |a_j| \left\| \mathbb{1}_{E_j} - \mathbb{1}_{\cup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)} < \varepsilon.$$

This shows functions of the form (i) are dense in $L^1(\mu)$.

It remains to prove (ii). Fix a function of the form (i),

$$\xi = \sum_{j=1}^{n} a_j \mathbb{1}_{\cup_{i=1}^{m_j} I_j^i}.$$

Let $\varepsilon > 0$. For any open interval I_j^i, there exists $f_j^i \in C_c(\mathbb{R})$ such that $\left| f_j^i - \mathbb{1}_{I_j} \right|_{L^1(\mu)} < \varepsilon$. Now

$$\|f_k - \mathbb{1}_I\|_{L^1(\mu)} = \int_{(a-\frac{1}{k},a)} f_k \mathrm{d}\mu + \int_{(b,b+\frac{1}{k})} f_k \mathrm{d}\mu \leq \mu\left(\left(a - \frac{1}{k}, a\right)\right) + \mu\left(\left(b, b + \frac{1}{k}\right)\right) \overset{k\to\infty}{\to} 0$$

by continuity from above, using the fact that Lebesgue-Stieltjes measures are locally bounded. Thus $\forall i, j \exists f_j^i \in C_c(\mathbb{R})$ such that

$$\left\| f_j^i - \mathbb{1}_{I_j^i} \right\|_{L^1(\mu)} < \frac{\varepsilon}{n \max_j |a_j| \max_j |m_j|}.$$

Therefore,

$$\left\| \sum_{j=1}^{n} a_j \mathbb{1}_{\cup_{i=1}^{m_j} I_j^i} - \sum_{j=1}^{n} a_j \left[\sum_{i=1}^{m_j} f_j^i \right] \right\|_{L^1(\mu)} \leq \sum_{j=1}^{n} |a_j| \sum_{i=1}^{m_j} \left\| \mathbb{1}_{I_j^i} - f_j^i \right\|_{L^1(\mu)} < \varepsilon.$$

□

Now we proceed with one more theorem on interchanging limits and integrals:

Theorem

Fix $a < b$. Consider $f : X \times [a,b] \to \mathbb{R}$ with $x,t) \mapsto f(x,t)$. Suppose

(i) $f(\cdot,t) \in L^1(\mu) \forall t \in [a,b]$

(ii) $\frac{\partial f}{\partial t}(x,t)$ exists $\forall (x,t) \in X \times [a,b]$

(iii) $\exists g \in L^1(\mu)$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x) \forall (x,t) \in X \times [a,b]$

Then $t \mapsto \int_X f(x,t) \mathrm{d}\mu(x)$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_X f(x,t) \mathrm{d}\mu(x) = \int_X \frac{\partial}{\partial t} f(x,t) \mathrm{d}\mu(x)$$

Proof: Fix $t_0 \in [a,b]$ and suppose $\{t_n\}_{n=1}^{\infty} \subseteq [a,b] \setminus \{t_0\}$ with $t_n \mapsto t_0$. By (ii),

$$\lim_{n\to\infty} \overbrace{\frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}}^{h_n(x)} = \frac{\partial f}{\partial t}(x,t_0).$$

So $\frac{\partial f}{\partial t}(\cdot,t)$ is measurable. By the Mean Value Theorem,

$$|h_n(x)| \le \sup_{t\in[a,b]} \left|\frac{\partial f}{\partial t}(x,t)\right| \overset{\text{(iii)}}{\le} g(x)$$

By the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{n\to\infty} \frac{\int f(x,t_n)\mathrm{d}\mu(x) - \int f(x,t_0)\mathrm{d}\mu(x)}{t_n - t_0} &= \lim_{n\to\infty} \int h_n(x)\mathrm{d}\mu(x) \\ &= \int \lim_{n\to\infty} h_n(x)\mathrm{d}\mu(x) \\ &= \int \frac{\partial f}{\partial t}(x,t_0)\mathrm{d}\mu(x). \end{aligned}$$

□

5 Convergence and Product Measures

5.A Modes of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space where $f_n, f : X \to \overline{\mathbb{R}}$ are measurable.

Recall from undergraduate analysis the idea of **uniform convergence**:

$$(1) \quad \sup_{x \in X} |f_n(x) - f(x)| \overset{n\to\infty}{\longrightarrow} 0$$

and **pointwise convergence**:

$$(2) \quad f_n(x) \overset{n\to\infty}{\longrightarrow} f(x) \forall x \in X.$$

We introduce the notion of **pointwise μ-a.e. convergence**:

$$(3) \quad f_n(x) \overset{n\to\infty}{\longrightarrow} f(x) \; \mu\text{-a.e. } \forall x \in X$$

and recall our prior notion of $L^1(\mu)$ convergence:

$$(4) \quad \|f_n - f\|_{L^1(\mu)} \overset{n\to\infty}{\longrightarrow} 0 \Longleftrightarrow \lim_{n\to+\infty} \int |f_n - f| \mathrm{d}\mu(x) = 0$$

We wish to understand exactly how these various **modes of convergence** are related. It is easy to see that (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3). The following examples illustrate that other relations are less clear.

Example: Splat

Let $f_n = \frac{1}{n} \mathbb{1}_{[0,n]}$ and $f = 0$. Then $\|f_n - f\|_{L^1(\mu)} = \|f_n\|_{L^1(\mu)} = 1 \nrightarrow 0$. But this converges uniformly, showing that (1) $\nRightarrow$ (4).

Example: Refining Wave

For $n \geq 1$, write $n = 2^k + j$ where $0 \leq j < 2^k$. Then define

$$f_n = \mathbb{1}_{[j/2^k, (j+1)/2^k]}.$$

Notice that f_n does not converge pointwise, since there it infinitely oscillates between 0 and 1, and $\|f_n\|_{L^1(\mu)} = \frac{1}{2^k} \to 0$, so it converges in $L^1(\mu)$. Thus shows that (4) $\nRightarrow$ (3).

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Thus, we need to modify our $L^1(\mu)$ convergence definition so that we can relate it to the other modes of convergence. Inspired by our work with the DCT, we might try adding a dominating function to (3). It turns out that with this addition, we will have (3) $\Rightarrow$ (4) In particular, suppose $\exists g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ μ-a.e. Then we can notice $|f_n(x) - f(x)| \leq 2g(x)$ μ-a.e., so by the DCT,

$$\lim_{n\to\infty} \int |f_n(x) - f(x)| \mathrm{d}\mu(x) = \int \lim_{n\to\infty} |f_n(x) - f(x)| \mathrm{d}\mu(x) = 0.$$

What about the other direction?

We will work to show (4) $\Rightarrow$ (3) “up to a subsequence”, by which we mean $\exists f_{n_k}$ such that $f_{n_k} \to f$ μ-a.e. Our strategy will invoke a new concept...

5.B Convergence in Measure

Definition: Converges / Cauchy in Measure

A sequence of measurable functions $f_n : X \to \mathbb{R}$ **converges in measure** to a limiting measurable function $f : X \to \mathbb{R}$ if $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Likewise, f_n is **Cauchy in measure** if $\forall \varepsilon > 0$,

$$\lim_{n,m\to\infty} \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

Remark

On homework, we will show that when $\mu(X) < +\infty$, convergence in measure is **metrizable**.

Proposition

In fact, for arbitrary $(X, \mathcal{M}, \mu)$, convergence in measure $\Rightarrow$ Cauchy in measure.

Proof: Fix $\varepsilon > 0$. Then

$$\{x : |f_n(x) - f_m(x)| \geq \varepsilon\} \subseteq \left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}$$

since otherwise $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, contradicting $|f_n(x) - f_m(x)| \geq \varepsilon$. Now by subadditivity,

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) \leq \mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}\right) + = \mu\left(\left\{x : |f_m(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}\right).$$

As $n, m \to +\infty$, the right hand side goes to zero, so f_n is Cauchy in measure.

$\square$

Theorem

Consider $f_n : X \to \mathbb{R}$ measurable.

(i) If f_n is Cauchy in measure, then $\exists f : X \to \mathbb{R}$ measurable such that
  (a) $f_n \to f$ in measure
  (b) $\exists f_{n_k}$ such that $f_{n_k} \to f$ μ-a.e.

(ii) If in addition $f_n \to g$ in measure, then $f = g$ μ-a.e.

Proof: First, we must find our guess for f. Since f_n is Cauchy in measure, $\exists f_{n_k}$ such that $\forall k \in \mathbb{N}$ we have

$$\mu\left(\underbrace{\left\{x : \left|\underbrace{f_{n_k}(x)}_{g_k(x)} - f_{n_{k+1}}(x)\right| \geq \frac{1}{2^k}\right\}}_{E_k}\right) \leq \frac{1}{2^k}.$$

By countable subadditivity, $\forall \ell \in \mathbb{N}$ we have

$$\mu\left(\bigcup_{k=\ell}^{\infty} E_k\right) \leq \sum_{k=\ell}^{\infty} \mu(E_k) \leq \sum_{k=\ell}^{\infty} \frac{1}{2^k} \leq \frac{1}{2^{\ell-1}}.$$

Define

$$F_\ell := \bigcup_{k=\ell}^{\infty} E_k \quad \text{and } G := \bigcup_{\ell=1}^{\infty} F_\ell^c.$$

Note that if $x \notin F_\ell \Leftrightarrow x \in \cap_{k=\ell}^{\infty} E_k$, then for $i \geq j \geq \ell$,

$$\left|g_i(x) - g_j(x)\right| \leq \sum_{k=j}^{i} \left|g_k(x) - g_{k+1}(x)\right| \leq \sum_{k=j}^{i} \frac{1}{2^k} \leq \frac{1}{2^{j-1}}.$$

Thus if $x \notin F_\ell$ for some $\ell \in \mathbb{N}$, then $x \in G$ and $\{g_i(x)\}_{i=1}^{\infty}$ is Cauchy, so it converges to $\lim_{i\to+\infty} g_i(x) \in \mathbb{R}$.

Now define

$$f(x) = \begin{cases} \lim_{i\to+\infty} g_i(x) & \text{if } x \in G \\ 0 & \text{if } x \in G^c \end{cases}$$

Further,

$$\mu(G^c) = \mu\left(\bigcap_{\ell=1}^{\infty} F_\ell\right) \leq \mu(F_\ell) \leq \frac{1}{2^{\ell-1}}$$

so $\mu(G^c) = 0$. So $f_{n_k} = g_k \to f$ μ-a.e.

This shows (i)(b). Now we show (i)(a).

If $x \in F_\ell$ and $j \geq \ell$ then $\left|f(x) - g_j(x)\right| = \lim_{i\to\infty} \left|g_i(x) - g_j(x)\right| \leq \frac{1}{2^{j-1}}$. Therefore by contraposition, if $\left|f(x) - g_j(x)\right| \geq \frac{1}{2^{j-1}}$ and $j \geq \ell$ we have $x \in F_\ell$.

Thus for all $\varepsilon > 0$ and $\ell \in \mathbb{N}$ large enough so that $\varepsilon > \frac{1}{2^{\ell-1}}$, if $j \geq \ell$, then

$$\mu\left(\left\{x : \left|f(x) - g_j(x)\right| \geq \varepsilon\right\}\right) \leq \mu\left(\left\{x : \left|f(x) - g_j(x)\right| > \frac{1}{2^{\ell-1}}\right\}\right) \leq \mu(F_\ell) \leq \frac{1}{2^{\ell-1}}$$

So $f_{n_j} = g_j \to f$ in measure. Now applying the same argument as before,

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq \left\{x : \left|f_n(x) - f_{n_j}(x)\right| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : \left|f_{n_j}(x) - f(x)\right| \geq \frac{\varepsilon}{2}\right\}$$

so

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \mu\left(\left\{x : \left|f_n(x) - f_{n_j}(x)\right| \geq \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : \left|f_{n_j}(x) - f(x)\right| \geq \frac{\varepsilon}{2}\right\}\right).$$

Sending $n, j \to +\infty$ shows $f_n \to f$ in measure.

Finally we show (ii). Suppose $f_n \to g$ in measure and fix $\varepsilon > 0$. By the same argument,

$$\mu(\{x : |f(x) - g(x)| \geq \varepsilon\}) \leq \mu\left(\left\{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}\right).$$

By (i) and the given, notice the right hand side goes to zero as $n \to +\infty$, and the left hand side is independent of n so

$$\mu(\{x : |g(x) - f(x)| \geq \varepsilon\}) = 0$$

and

$$\mu(\{x : |g(x) - f(x)| > 0\}) = \mu\left(\bigcup_{k=1}^{\infty}\left\{x : |g(x) - f(x)| \geq \frac{1}{k}\right\}\right) \leq \sum_{k=1}^{\infty}\mu\left(\left\{x : |g(x) - f(x)| \geq \frac{1}{k}\right\}\right) = 0$$

so $f = g$ μ-a.e.

□

We now apply these results to study convergence in $L^1(\mu)$.

Proposition

(i) If f_n is Cauchy in $L^1(\mu)$, then it's Cauchy in measure.
(ii) If f_n is convergent in $L^1(\mu)$, then it's convergent in measure.

Proof: If f_n is convergent in $L^1(\mu)$, because $L^1(\mu)$ is a metric space, we have that f_n is Cauchy. Then (i) from the previous proposition implies f_n is Cauchy in measure.

Define $E_{n,m,\varepsilon} := \{x : |f_n(x) - f_m(x)| \geq \varepsilon\}$. Then

$$\varepsilon\mu(E_{n,m,\varepsilon}) = \int_{E_{n,m,\varepsilon}} \varepsilon \,\mathrm{d}\mu \leq \int_{E_{n,m,\varepsilon}} |f_n(x) - f_m(x)| \,\mathrm{d}\mu(x) \leq \int_X |f_n(x) - f_m(x)| \,\mathrm{d}\mu(x) = \|f_n(x) - f_m(x)\|_{L^1(\mu)}.$$

Since f_n is Cauchy in $L^1(\mu)$, $n, m \to +\infty$ means the right hand side goes to zero. So f_n is convergent in measure.

□

Corollary

If f_n is Cauchy in $L^1(\mu)$, then $f \in L^1(\mu)$ and a subsequence f_{n_k} such that $f_{n_k} \to f$ μ-a.e.

Proof: By the previous theorem, f_n is Cauchy in measure. By previous proposition, $\exists f : X \to \mathbb{R}$ measurable such that $f_{n_k} \to f$ μ-a.e. It remains to show $f \in L^1(\mu)$. By Fatou's Lemma,

$$\int |f| \,\mathrm{d}\mu = \int \liminf_{k \to +\infty} |f_{n_k}| \,\mathrm{d}\mu \leq \liminf_{k \to +\infty} \int |f_{n_k}| \,\mathrm{d}\mu = \liminf_{k \to +\infty} \|f_{n_k}\|_{L^1(\mu)} < +\infty$$

□

Corollary

$L^1(\mu)$ is a Banach space, that is, a complete normed vector space.

Proof: Let f_n be Cauchy in $L^1(\mu)$. By the previous corollary, $\exists f \in L^1(\mu)$ and a subsequence f_{n_k} such that $f_{n_k} \to f$ μ-a.e. By Fatou's Lemma,

$$\int \left|f_{n_k} - f\right| \mathrm{d}\mu = \int \liminf_{j\to\infty} \left|f_{n_k} - f_{n_j}\right| \mathrm{d}\mu \le \liminf_{j\to\infty} \left\|f_{n_k} - f_{n_j}\right\|_{L^1(\mu)}.$$

Thus the left hand side goes to zero as $k \to +\infty$ so $f_{n_k} \to f$ in $L^1(\mu)$. Finally, on any metric space, if a subsequence of a Cauchy sequence converges to a limit, then the whole sequence must converge to the same limit.

□

So to summarize the modes of convergence:

$$f_n \to f \text{ in } L^1(\mu) \Rightarrow f_n \Rightarrow f \text{ in measure} \Rightarrow f_{n_k} \overset{\text{up to subseq}}{\Rightarrow} f \ \mu\text{-a.e.}$$

5.C Product Measures

Lecture 14 Nov 13

Notation

Let
- $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha\in A}$ a countable collection of measurable spaces
- $X = \prod_{\alpha\in A} X_\alpha$
- Let π_α denote the projection of X onto X_α so $\pi_\alpha : X \to X_\alpha$

Definition

The product σ-algebra is

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}\right)$$

For example, if each $E_\alpha = [a, b] \subseteq \mathbb{R}$, then we define a rectanglar prism.

Our first goal is to show $\bigotimes_{i=1}^d \mathcal{B}_\mathbb{R} = \mathcal{B}_{\mathbb{R}^d}$.

Proposition

Given $\mathcal{E}_\alpha \subseteq 2^{X_\alpha}$ such that $X_\alpha \in \mathcal{E}_\alpha$ and $\mathcal{M}_\alpha = \mathcal{M}(\mathcal{E}_\alpha)$ then

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha = \mathcal{M}\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right\}\right)$$

Proof: HW7

□

Recall the following facts, which will be useful in the following the following theorem:

(i) $\{X_i\}_{i=1}^n$ separable $\Rightarrow X = \prod_{i=1}^n X_i$ separable

(ii) In a separable metric space, every open set can be written as a countable union of open balls

(iii) $\{E_i\}_{i=1}^n$ open $\Rightarrow \prod_{i=1}^n E_i$ open

Theorem

Given metric spaces $X_1, X_2, ..., X$ and $\prod_{i=1}^d X_i$ endowed with the metric $d_{\max}((x_1, ..., x_d), (y_1, ..., y_d)) = \max_{1 \le i \le d} d_i(x_i, y_i)$. Then $\bigotimes_{i=1}^d \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$. Further, if $\{X_i\}_{i=1}^d$ is separable, then $\mathcal{B}_X = \otimes_{i=1}^d \mathcal{B}_{X_i}$.

Proof: By the proposition,

$$\bigotimes_{i=1}^d \mathcal{B}_{X_i} = \mathcal{M}\left(\left\{\prod_{i=1}^n E_i : E_i \subseteq X_i\right\}\right) \subseteq \mathcal{M}(\{U : U \subseteq X \text{ open}\}) = \mathcal{B}_X.$$

Since X is endowed with $d_{\max}$, we know $B = \prod_{i=1}^d B_i$ for balls $B_i \subseteq X_i$.

□

Remark

$d_{\max}$ is convenient because $\mathcal{B}_r(x_1, ..., x_n) = \prod_{i=1}^d \mathcal{B}_r(x_i)$.

Remark

Since the definition of $\mathcal{B}_X$ depends only on the topology of X, the result continues to hold if X is endowed with any equivalent metric.

Suppose we have measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ and rectangles $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$. Then $\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\})$. Our goal is to prove the existnce of a unique measure ω on the measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ with the property

$$\omega(A \times B) = \mu(A)\nu(B) \forall A \in \mathcal{M}, B \in \mathcal{N}.$$

We will denote this measure by $\mu \otimes \nu := \omega$. To accomplish this goal, we will use the **Monotone Class Theorem**.

Definition: Monotone Class

$C \subseteq 2^X$ is called a **monotone class** if it is a nonempty collection such that

(i) $\{E_i\}_{i=1}^\infty \subseteq \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \cdots \Rightarrow \cup_{i=1}^\infty E_i \in \mathcal{C}$.

(ii) $\{E_i\}_{i=1}^\infty \subseteq \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \cdots \Rightarrow \cap_{i=1}^\infty E_i \in \mathcal{C}$.

Proposition

Given any $\mathcal{E} \subseteq 2^X$ nonempty, $\exists$ a smallest monotone class containing $\mathcal{E}$ denoted $\mathcal{C}(\mathcal{E})$.

Proof: HW

□

Definition: X-section and Y-section

For any $E \in \mathcal{M} \otimes \mathcal{N}$, define the **x-section** E_x and **y-section** E_y by

$$E_x = \{y : (x, y) \in E\}$$
$$E_y = \{x : (x, y) \in E\}$$

Here are some basic properties of sections:

(i) If $E = A \times B, A \in \mathcal{M}, B \in \mathcal{N}$ then

$$E_x = \begin{cases} B \text{ if } x \in A \\ \emptyset \text{ if } x \notin A \end{cases}$$

$$E_y = \begin{cases} A \text{ if } y \in B \\ \emptyset \text{ if } y \notin B \end{cases}$$

(ii) Given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M} \otimes \mathcal{N}$,

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_X = \left\{y : (x, y) \in \bigcup_{i=1}^{\infty} E_i\right\}$$
$$= \bigcup_{i=1}^{\infty} \{y : (x, y) \in E_i\}$$
$$= \bigcup_{i=1}^{\infty} (E_i)_X$$

(iii) Given $E \in \mathcal{M} \otimes \mathcal{N}$,

$$(E^c)_X = \{y : (x, y) \in E^c\}$$
$$= \{y : (x, y) \in E\}^c$$
$$= (E_x)^c.$$

(iv) If $E \in \mathcal{M} \otimes \mathcal{N}$ then

$$\nu(E_x) = \int_Y \mathbb{1}_{E_x}(y)\, \mathrm{d}\nu(y) = \int_Y \mathbb{1}_E(x, y)\, \mathrm{d}\nu(y)$$

Proposition

If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E_x \in \mathcal{N}, E^y \in \mathcal{M}$, then $\forall x \in X, y \in Y$.

Proof: Let $\mathcal{E} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{above holds}\}$. By a, $\mathcal{E}$ contains all rectangles. By b and c, $\mathcal{E}$ is a σ-algebra.

By definition, $\mathcal{M} \otimes \mathcal{N}$ is the smallest σ-algebra containing all rectangles. Thus $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$.

□

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